Matematika vzvratne vožnje vozila s priklopniki

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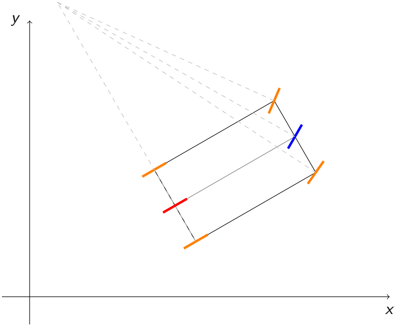
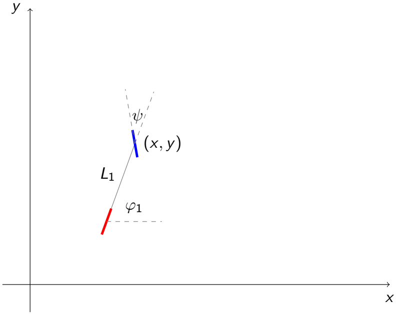
21.9.2019

Vozila s priklopniki



Vozila s priklopniki



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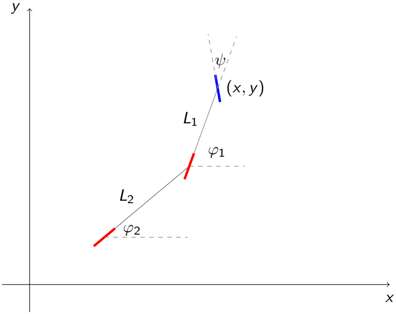
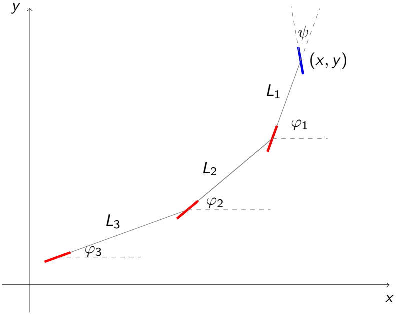


Vozila s priklopniki



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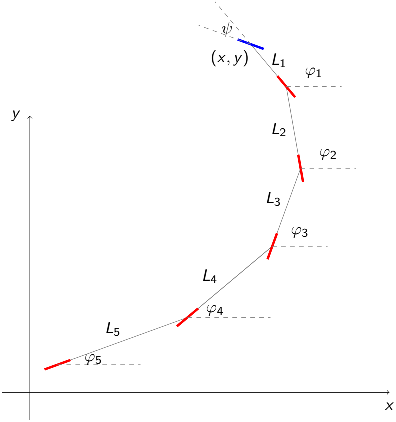
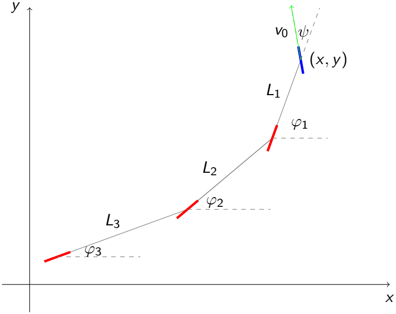


Gibanje vozila s priklopniki



Krmiljenje je določeno z:

- $\psi(t)$ ... kot zasuka prednjega kolesa
- $v_0(t)$ ... hitrost prednjega kolesa

Gibanje vozila s priklopniki

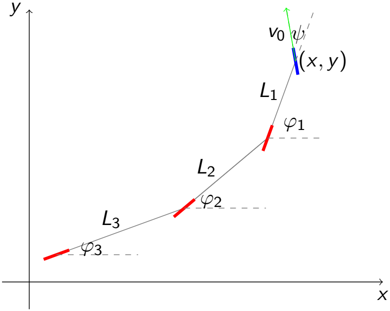
V kratkem času dt:

$$dx = v_0\, dt \cos(\varphi_1 + \psi)$$

$$dy = v_0\, dt \sin(\varphi_1 + \psi)$$

od koder dobimo (delimo z dt):

$$\begin{aligned} \dot{x} &= v_0 \cos(\varphi_1 + \psi) \\ \dot{y} &= v_0 \sin(\varphi_1 + \psi) \end{aligned}$$



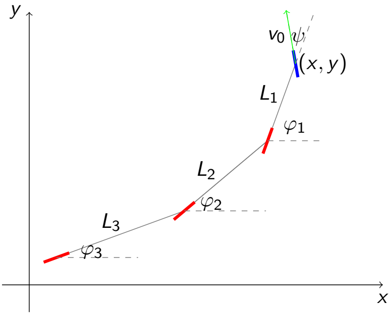
Gibanje vozila s priklopniki

V kratkem času dt:

$$d\varphi_1 = v_0\, dt \sin(\psi) \frac{1}{L_1}\,,$$

od koder dobimo:

$$\dot{\varphi}_1 = \frac{v_0}{L_1} \sin(\psi)\,.$$



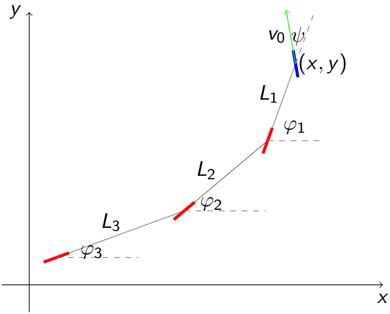
Gibanje vozila s priklopniki

V kratkem času dt:

$$d\varphi_2 = v_0\, dt \cos(\psi) \sin(\varphi_1 - \varphi_2) \frac{1}{L_2}\,,$$

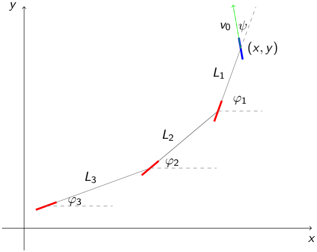
od koder dobimo:

$$\dot{\varphi}_2 = \frac{v_0}{L_2} \cos(\psi) \sin(\varphi_1 - \varphi_2)\,.$$



Gibanje vozila s priklopniki

$$
\begin{aligned}
\dot{x} &= v_0 \cos(\varphi_1 + \psi) \\
\dot{y} &= v_0 \sin(\varphi_1 + \psi) \\
\dot{\varphi}_1 &= \frac{v_0}{L_1} \sin(\psi) \\
\dot{\varphi}_2 &= \frac{v_0}{L_2} \cos(\psi) \sin(\varphi_1 - \varphi_2) \\
\dot{\varphi}_3 &= \frac{v_0}{L_3} \cos(\psi) \cos(\varphi_1 - \varphi_2) \sin(\varphi_2 - \varphi_3) \\
&\vdots \\
\dot{\varphi}_n &= \frac{v_0}{L_n} \cos(\psi) \cos(\varphi_1 - \varphi_2) \cdots \cos(\varphi_{n-2} - \varphi_{n-1}) \sin(\varphi_{n-1} - \varphi_n) .
\end{aligned}
$$



PRIMERI

- Vožnja naprej.
- Vožnja nazaj.
- Vožnja naprej ($y = 0$ in $\varphi_1 = 0$).
- Vožnja nazaj ($y = 0$ in $\varphi_1 = 0$). Več primerov, ko $\varphi_2(0), \varphi_3(0), \ldots \to 0$.

Linearnih sistemi

Naj funkcija $x(t)$ zadošča linearni diferencialni enačbi

$$\dot{x}(t) = a\,x(t) \qquad x(0) = x_0\,,$$

kjer je $a \in \mathbb{R}$.
Kako rešiti to enačbo?
Imamo nastavek $x(t) = e^{\lambda t}$. Od to je $\dot{x}(t) = \lambda e^{\lambda t}$, kar vstavimo v enačbo in dobimo

$$\lambda e^{\lambda t} = a\,e^{\lambda t} \Rightarrow \lambda = a \Rightarrow x(t) = x_0 e^{at}\,.$$

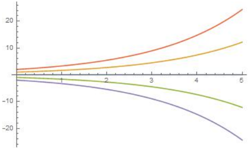
Linearni sistemi

Rešitve sistema $\dot{x}(t) = a\,x(t)$, $x(0) = x_0$ so oblike

$$x(t) = x_0 e^{at}\,.$$

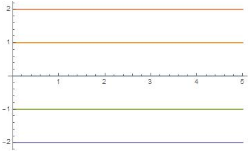
$a > 0$

$\lim_{t\to\infty} x(t) = \pm\infty$



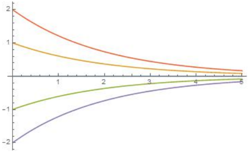
$a = 0$

$x(t) =$konts.



$a < 0$

$\lim_{t\to\infty} x(t) = 0$



Linearni sistemi

Naj vektorska funkcija $\vec{x}(t)$ zadošča sistemu linearnih diferencialnh enačb (v matrični obliki)

$$\dot{\vec{x}}(t) = A\,\vec{x}(t) \qquad \vec{x}(0) = \vec{x}_0\,,$$

kjer je $A \in \mathbb{R}^{n\times n}$.
Kako rešiti to enačbo?
Imamo nastavek $\vec{x}(t) = e^{\lambda t}\vec{c}$. Od to je $\dot{\vec{x}}(t) = \lambda e^{\lambda t}\vec{c}$, kar vstavimo v enačbo in dobimo

$$\lambda e^{\lambda t}\vec{c} = e^{\lambda t}A\vec{c} \Rightarrow (A - \lambda \mathrm{Id})\vec{c} = 0\,.$$

Linearni sistemi

Dobimo pogoj

$$(A - \lambda \mathrm{Id})\vec{c} = 0\,,$$

kar pomeni, da je λ lastna vrednost matrika A, $\vec{c}$ pa lastni vektor matrike A.
Če so $\lambda_1, ..., \lambda_n$ različne lastne vrednosti matrike A in $\vec{c}_1, ..., \vec{c}_n$ pripadajoči lastni vektorji, je splošna rešitev oblike

$$\vec{x}(t) = a_1 e^{\lambda_1 t}\vec{c}_1 + a_2 e^{\lambda_2 t}\vec{c}_2 + ... + a_n e^{\lambda_n t}\vec{c}_n \qquad a_i \in \mathbb{R}\,.$$

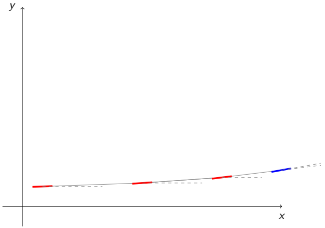
Naj bodo vse lastne vrednosti λ_i matrike A negativne ($\lambda_i < 0$). Potem velja

$$\lim_{t\to\infty} \vec{x}(t) = 0\,.$$

Gibanje vozila s priklopniki v smeri x-osi

Zanima nas vožnja v smeri x-osi, kjer so vse količine $y, \psi, \varphi_1, ..., \varphi_n \cong 0$. Linearizirajno spodnji sistem:

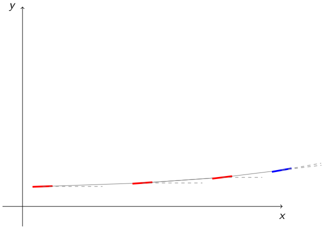
$$\begin{aligned}
\dot{x} &= v_0 \cos(\varphi_1 + \psi) \\
\dot{y} &= v_0 \sin(\varphi_1 + \psi) \\
\dot{\varphi}_1 &= \frac{v_0}{L_1} \sin(\psi) \\
\dot{\varphi}_2 &= \frac{v_0}{L_2} \cos(\psi) \sin(\varphi_1 - \varphi_2) \\
\dot{\varphi}_3 &= \frac{v_0}{L_3} \cos(\psi) \cos(\varphi_1 - \varphi_2) \sin(\varphi_2 - \varphi_3) \\
&\vdots \\
\dot{\varphi}_n &= \frac{v_0}{L_n} \cos(\psi) \cos(\varphi_1 - \varphi_2) \cdots \cos(\varphi_{n-2} - \varphi_{n-1}) \sin(\varphi_{n-1} - \varphi_n) .
\end{aligned}$$



Gibanje vozila s priklopniki v smeri x-osi

Dobimo

$$
\begin{aligned}
\dot{x} &= v_0 \\
\dot{y} &= v_0(\varphi_1 + \psi) \\
\dot{\varphi}_1 &= \frac{v_0}{L_1}\psi \\
\dot{\varphi}_2 &= \frac{v_0}{L_2}(\varphi_1 - \varphi_2) \\
\dot{\varphi}_3 &= \frac{v_0}{L_3}(\varphi_2 - \varphi_3) \\
&\vdots \\
\dot{\varphi}_n &= \frac{v_0}{L_n}(\varphi_{n-1} - \varphi_n)\,.
\end{aligned}
$$



Prva enačba $\dot{x} = v_0$ je trivialno rešljiva v $x(t) = v_0\, t + x_0$.

Gibanje vozila s priklopniki v smeri x-osi **BREZ VODENJA**

Oglejmo si primer gibanja, ko prvo vozilo vozi naravnost v smeri x-osi:

$$y(t) \equiv 0 \qquad \varphi_1(t) \equiv 0 \qquad \psi(t) \equiv 0\,.$$

Dobimo:

$$\begin{aligned}
\dot{\varphi}_2 &= -\frac{v_0}{L_2}\varphi_2 \\
\dot{\varphi}_3 &= \frac{v_0}{L_3}(\varphi_2 - \varphi_3) \\
&\vdots \\
\dot{\varphi}_n &= \frac{v_0}{L_n}(\varphi_{n-1} - \varphi_n)\,.
\end{aligned}$$

$$\begin{bmatrix} \dot{\varphi}_2 \\ \vdots \\ \dot{\varphi}_n \end{bmatrix} = \begin{bmatrix} -\frac{v_0}{L_2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & \frac{v_0}{L_n} & -\frac{v_0}{L_n} \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}.$$

Gibanje vozila s priklopniki v smeri x-osi **BREZ VODENJA**

Lastne vrednosti matrike

$$\begin{bmatrix} -\frac{v_0}{L_2} & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & \frac{v_0}{L_n} & -\frac{v_0}{L_n} \end{bmatrix}$$

dobimo kot rešitve enačbe

$$\det \begin{bmatrix} -\frac{v_0}{L_2} - \lambda & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & \frac{v_0}{L_n} & -\frac{v_0}{L_n} - \lambda \end{bmatrix} = 0\,.$$

Gibanje vozila s priklopniki v smeri x-osi **BREZ VODENJA**

$$\det \begin{bmatrix} -\frac{v_0}{L_2} - \lambda & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & \cdots & \frac{v_0}{L_n} & -\frac{v_0}{L_n} - \lambda \end{bmatrix} =$$

$$= (-\frac{v_0}{L_2} - \lambda) \cdots (-\frac{v_0}{L_n} - \lambda) = 0$$

in dobimo lastne vrednosti

$$\lambda_1 = -\frac{v_0}{L_2} \quad \cdots \quad \lambda_{n-1} = -\frac{v_0}{L_n}\,.$$

Če $v_0 > 0$ so vse lastne vrednosti negativne, $\lim_{t \to \infty} \varphi_i(t) = 0$.

Če $v_0 < 0$ so vse lastne vrednosti pozitivne, $\lim_{t \to \infty} \varphi_i(t) = \pm\infty$.

Stabilno vodenje vozila s priklopniki v smeri x-osi

Naj bo $v_0 < 0$. Iščemo $\psi(t)$, da bo veljalo

$$\lim_{t\to\infty} y(t) = 0 \quad \lim_{t\to\infty} \varphi_i(t) = 0 \quad \forall i = 1, \ldots, n\,.$$

Imamo

$$\begin{aligned}
\dot{y} &= v_0(\psi + \varphi_1) \\
\dot{\varphi}_1 &= \frac{v_0}{L_1}\psi \\
\dot{\varphi}_2 &= \frac{v_0}{L_2}(\varphi_1 - \varphi_2) \\
\dot{\varphi}_3 &= \frac{v_0}{L_3}(\varphi_2 - \varphi_3) \\
&\vdots \\
\dot{\varphi}_n &= \frac{v_0}{L_n}(\varphi_{n-1} - \varphi_n)\,.
\end{aligned}$$

Stabilno vodenje vozila s priklopniki v smeri x-osi

Ali obstajajo konstante $\alpha_0, \alpha_1, ..., \alpha_n$, da bo vodenje oblike

$$\psi = \alpha_0 y + \alpha_1 \varphi_1 + \cdots \alpha_n \varphi_n$$

peljalo vozilo vzvratno v smeri x-osi?

Stabilno vodenje vozila s priklopniki v smeri x-osi

Dobimo linearni sistem enačb:

$$\begin{bmatrix} \dot{y} \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \vdots \\ \dot{\varphi}_n \end{bmatrix} = \begin{bmatrix} v_0\alpha_0 & v_0(1+\alpha_1) & v_0\alpha_2 & \cdots & v_0\alpha_n \\ \frac{v_0}{L_1}\alpha_0 & \frac{v_0}{L_1}\alpha_1 & \frac{v_0}{L_1}\alpha_2 & \cdots & \frac{v_0}{L_1}\alpha_n \\ 0 & \frac{v_0}{L_2} & -\frac{v_0}{L_2} & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & \frac{v_0}{L_n} & -\frac{v_0}{L_n} \end{bmatrix} \begin{bmatrix} y \\ \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix}.$$

Hočemo določiti $\alpha_0, \alpha_1, ..., \alpha_n$ tako, da bodo vse lastne vrednosti zgornje matrike negativne.

Naj bo $\lambda < 0$. Poiščimo pogoj, ki mu morajo zadoščati $\alpha_0, ..., \alpha_n$, da bo λ lastna vrednost matrike:

$$\begin{bmatrix} v_0\alpha_0 & v_0(1+\alpha_1) & v_0\alpha_2 & \cdots & v_0\alpha_n \\ \frac{v_0}{L_1}\alpha_0 & \frac{v_0}{L_1}\alpha_1 & \frac{v_0}{L_1}\alpha_2 & \cdots & \frac{v_0}{L_1}\alpha_n \\ 0 & \frac{v_0}{L_2} & -\frac{v_0}{L_2} & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & \frac{v_0}{L_n} & -\frac{v_0}{L_n} \end{bmatrix}.$$

Naj bo $\begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix}$ lastni vektor, ki pripada lastni vrednosti λ:

$$\begin{bmatrix} v_0\alpha_0 & v_0(1+\alpha_1) & v_0\alpha_2 & \cdots & v_0\alpha_n \\ \frac{v_0}{L_1}\alpha_0 & \frac{v_0}{L_1}\alpha_1 & \frac{v_0}{L_1}\alpha_2 & \cdots & \frac{v_0}{L_1}\alpha_n \\ 0 & \frac{v_0}{L_2} & -\frac{v_0}{L_2} & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & \frac{v_0}{L_n} & -\frac{v_0}{L_n} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Začnimo v spodnji vrstici:

$$x_{n-1} = x_n \left(1 + \frac{\lambda L_n}{v_0}\right) .$$

Nato

$$x_{n-2} = x_{n-1} \left(1 + \frac{\lambda L_{n-1}}{v_0}\right) ,$$

$$x_{n-3} = x_{n-2} \left(1 + \frac{\lambda L_{n-2}}{v_0}\right) ,$$

$$\vdots$$

$$x_1 = x_2 \left(1 + \frac{\lambda L_2}{v_0}\right) .$$

Prvi dve vrstici nam data:

$$v_0\alpha_0 x_0 + v_0 x_1 + v_0\alpha_1 x_1 + v_0\alpha_2 x_2 + ... + v_0\alpha_n x_n = \lambda x_0$$
$$v_0\alpha_0 x_0 + v_0\alpha_1 x_1 + v_0\alpha_2 x_2 + ... + v_0\alpha_n x_n = L_1\lambda x_1 .$$

Enačbi odštejemo in dobimo

$$x_0 = x_1 \left(1 + \frac{\lambda L_1}{v_0}\right) \frac{v_0}{\lambda} .$$

Torej, če predpostavimo, da je $\lambda < 0$ lastna vrednost, potem je

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \left(1+\frac{\lambda L_n}{v_0}\right)\cdots\left(1+\frac{\lambda L_1}{v_0}\right)\frac{v_0}{\lambda} \\ \left(1+\frac{\lambda L_n}{v_0}\right)\cdots\left(1+\frac{\lambda L_2}{v_0}\right) \\ \vdots \\ \left(1+\frac{\lambda L_n}{v_0}\right)\left(1+\frac{\lambda L_{n-1}}{v_0}\right) \\ \left(1+\frac{\lambda L_n}{v_0}\right) \\ 1 \end{bmatrix}$$

pripadajoči lastni vektor, pogoj za $\alpha_0, ..., \alpha_n$ pa se glasi

$$x_0\alpha_0 + x_1\alpha_1 + ... + x_n\alpha_n = \frac{L_1\lambda}{v_0}x_1\,.$$

Naj bodo $\lambda_0, ..., \lambda_n < 0$. Za vsak λ_i dobimo pogoj oblike

$$x_0\alpha_0 + x_1\alpha_1 + ... + x_n\alpha_n = \frac{L_1\lambda_i}{v_0}x_1 \,,$$

kar nam da linearnen sistem $(n+1)$-enačb za $(n+1)$-neznak $\alpha_0, ..., \alpha_n$.

Pogoje za $\alpha_0, ..., \alpha_n$ zapišimo v matrični obliki:

$$\begin{bmatrix} x_0^{\lambda_0} & x_1^{\lambda_0} & \cdots & x_n^{\lambda_0} \\ x_0^{\lambda_1} & x_1^{\lambda_1} & \cdots & x_n^{\lambda_1} \\ \vdots & \vdots & \cdots & \vdots \\ x_0^{\lambda_n} & x_1^{\lambda_n} & \cdots & x_n^{\lambda_n} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} \frac{L_1 \lambda_0}{v_0} x_1^{\lambda_0} \\ \frac{L_1 \lambda_1}{v_0} x_1^{\lambda_1} \\ \vdots \\ \frac{L_1 \lambda_n}{v_0} x_1^{\lambda_n} \end{bmatrix}.$$

Pokažimo sedaj, da ima zgornja matrika neničelno determinanto, kar bo pomenilo, da je sistem enačb enolično rešljiv.

Matrika je oblike (transponirana):

$$\begin{bmatrix} \left(1+\frac{\lambda_0 L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_0 L_1}{v_0}\right)\frac{v_0}{\lambda_0} & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_n L_1}{v_0}\right)\frac{v_0}{\lambda_n} \\ \left(1+\frac{\lambda_0 L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_0 L_2}{v_0}\right) & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_n L_2}{v_0}\right) \\ \vdots & \cdots & \vdots \\ \left(1+\frac{\lambda_0 L_n}{v_0}\right)\left(1+\frac{\lambda_0 L_{n-1}}{v_0}\right) & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right)\left(1+\frac{\lambda_n L_{n-1}}{v_0}\right) \\ \left(1+\frac{\lambda_0 L_n}{v_0}\right) & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right) \\ 1 & \cdots & 1 \end{bmatrix}.$$

Pomnožimo drugo vrstico z $-\left(1+\frac{\lambda_0 L_1}{v_0}\right)\frac{v_0}{\lambda_0}$ in jo prištejmo prvi. Tretjo vrstico pomnožimo z $-\left(1+\frac{\lambda_0 L_2}{v_0}\right)$ in jo prištejemo drugi. S tem nadaljujemo, nato pa zadnjo vrstico pomnožimo z $-\left(1+\frac{\lambda_0 L_n}{v_0}\right)$ in jo prištejemo predzadnji. Nato drugi stolpec pomnožimo z $\frac{1}{\lambda_1-\lambda_0}$, tretji z $\frac{1}{\lambda_1-\lambda_0}$, in nazadnje zadnji stolpec z $\frac{1}{\lambda_1-\lambda_0}$. Nato prvo vrstico pomnožimo z $-\lambda_0$, drugo z $\frac{v_0}{L_2}$, tretjo z $\frac{v_0}{L_3}$ in tako naprej. Dobimo:

$$
\begin{bmatrix}
0 & \left(1+\frac{\lambda_1 L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_1 L_2}{v_0}\right)\frac{v_0}{\lambda_1} & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_n L_2}{v_0}\right)\frac{v_0}{\lambda_n} \\
0 & \left(1+\frac{\lambda_1 L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_1 L_3}{v_0}\right) & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right)\cdots\left(1+\frac{\lambda_n L_3}{v_0}\right) \\
\vdots & \vdots & \cdots & \vdots \\
0 & \left(1+\frac{\lambda_1 L_n}{v_0}\right)\left(1+\frac{\lambda_1 L_{n-1}}{v_0}\right) & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right)\left(1+\frac{\lambda_n L_{n-1}}{v_0}\right) \\
0 & \left(1+\frac{\lambda_1 L_n}{v_0}\right) & \cdots & \left(1+\frac{\lambda_n L_n}{v_0}\right) \\
0 & 1 & \cdots & 1 \\
1 & \frac{1}{\lambda_1-\lambda_0} & \cdots & \frac{1}{\lambda_n-\lambda_0}
\end{bmatrix}.
$$

Determinanta te matrike je do predznaka enaka determinanti zgornje desne podmatrike. Ponovljanje prejšnjega postopka nas končno privede do:

$$\begin{bmatrix} \left(1 + \frac{\lambda_{n-1} L_n}{v_0}\right) \frac{v_0}{\lambda_{n-1}} & \left(1 + \frac{\lambda_n L_n}{v_0}\right) \frac{v_0}{\lambda_n} \\ 1 & 1 \end{bmatrix}$$

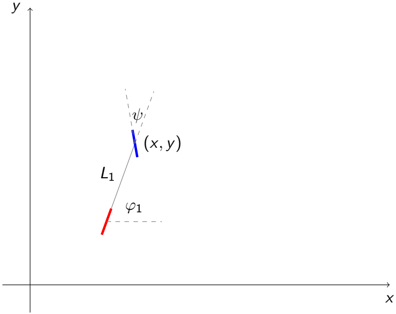
z neničelno determinanto $v_0 \left(\frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n} \right)$.

PRIMERI

Vozila s priklopniki $n = 1$

$$v_0 = -50 \qquad \lambda_0 = -1.0$$
$$L_1 = 100 \qquad \lambda_1 = -0.2$$

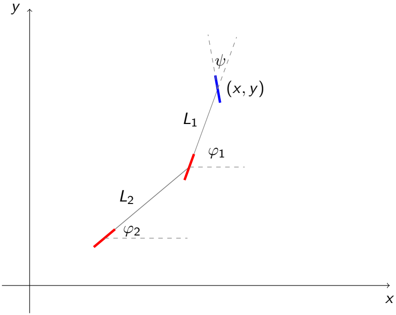
$$\psi = -0.008\, y + 3.2\, \varphi_1$$



Vozila s priklopniki $n = 2$

$$v_0 = -50 \qquad \lambda_0 = -1.0$$
$$L_1 = 100 \qquad \lambda_1 = -0.2$$
$$L_2 = 100 \qquad \lambda_2 = -0.3$$

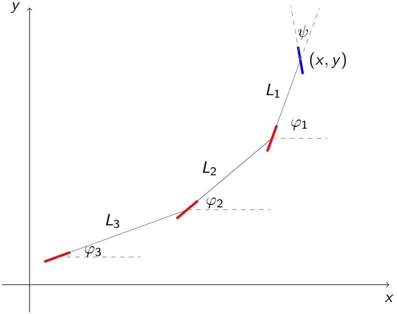
$$\psi = 0.0048\, y + 3.52\, \varphi_1 - 6.72\, \varphi_2$$



Vozila s priklopniki $n = 3$

$$\begin{aligned} v_0 &= -50 & \lambda_0 &= -1.0 \\ L_1 &= 100 & \lambda_1 &= -0.2 \\ L_2 &= 50 & \lambda_2 &= -0.3 \\ L_3 &= 100 & \lambda_3 &= -0.4 \end{aligned}$$

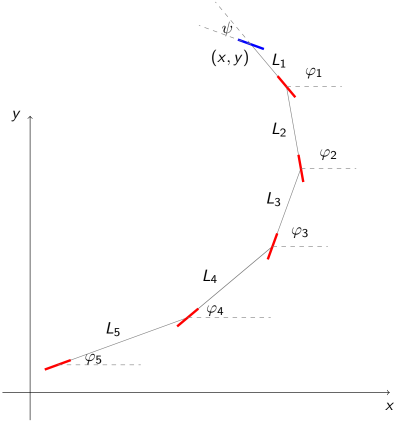
$$\psi = -0.00192\, y + 6.992\, \varphi_1 - 11.424\, \varphi_2 + 6.048\, \varphi_3$$



Vozila s priklopniki $n = 5$

$$
\begin{aligned}
v_0 &= -50 & \lambda_0 &= -1.0 \\
L_1 &= 80 & \lambda_1 &= -0.2 \\
L_2 &= 40 & \lambda_2 &= -0.3 \\
L_3 &= 80 & \lambda_3 &= -0.4 \\
L_4 &= 40 & \lambda_4 &= -0.5 \\
L_5 &= 80 & \lambda_5 &= -0.6
\end{aligned}
$$

$\psi = -0.00038\, y + 10.83\, \varphi_1 - 30.43\, \varphi_2 + 61.97\, \varphi_3 - 49.30\, \varphi_4 + 7.35\, \varphi_5$



Hvala za pozornost!