Diskretna verižnica

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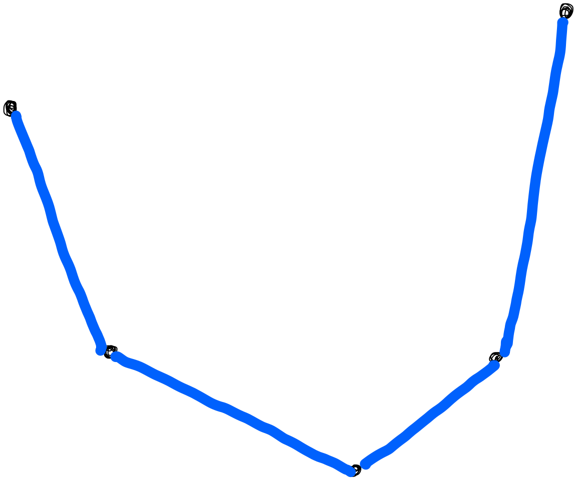
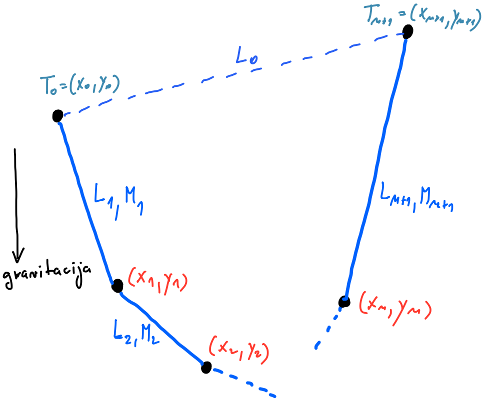
Naloga

- Danih je $n + 1$ homogenih votlih členkov (tulcev) dolžin L_i in mas M_i, $i = 1, 2, \ldots, n + 1$.
- Skozi členke potegnemo vrvico in jih stisnemo, da se dotikajo drug drugega.
- Vrvico pritrdimo na začetku prvega in na koncu zadnjega členka.
- Sistem spustimo, da prosto visi.
- Kakšno obliko zavzame?



Matematični opis

- Prvi tulec je pripet v $\boldsymbol{T}_0 = (x_0, y_0)$, zadnji v $\boldsymbol{T}_{n+1} = (x_{n+1}, y_{n+1})$.
- Ostala stičišča tulcev v (x_i, y_i), $i = 1, 2, \ldots, n$, so neznanke.
- Definirajmo še $L_0 = \text{razdalja}(\boldsymbol{T}_0, \boldsymbol{T}_{n+1})$.



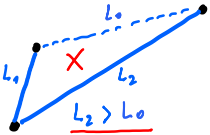
Trditev (poligonska neenakost)

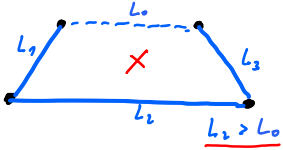
Verižnica obstaja natanko tedaj, ko je

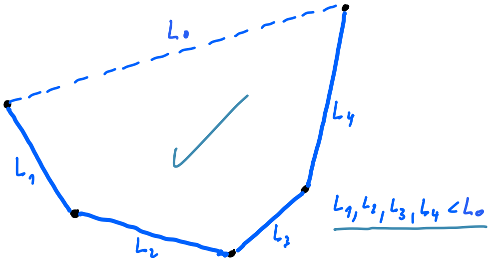
$$L_i < \sum_{j=0,\, j\neq i}^{n+1} L_j, \quad i = 0, 1, \ldots, n+1.$$

Lema

Če je $L_i < L_0$, $i = 1, 2, \ldots, n+1$, *verižnica predstavlja graf odsekoma linearne konveksne funkcije.*







- Omejimo se na primer iz leme.

- Neznanke: $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$.
- Določili jih bomo tako, da bomo minimizirali skalirano potencialno energijo

$$F(\boldsymbol{x}, \boldsymbol{y}) = \sum_{j=1}^{n+1} M_i \frac{y_{i-1} + y_i}{2}.$$

- Pogoji:

$$(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 - L_i^2 = 0, \quad i = 1, 2, \ldots, n+1. \tag{1}$$

- Iščemo minimum:

$$\min_{\boldsymbol{x}, \boldsymbol{y}} F(\boldsymbol{x}, \boldsymbol{y}), \quad \text{pri pogojih (1)}.$$

- Vezani ekstrem.

- Uvedemo Lagrangeeve multiplikatorje $\boldsymbol{\lambda} = (\lambda_i)_{i=1}^{n+1}$.
- Iščemo "navadni" ekstrem (minimum) funkcije

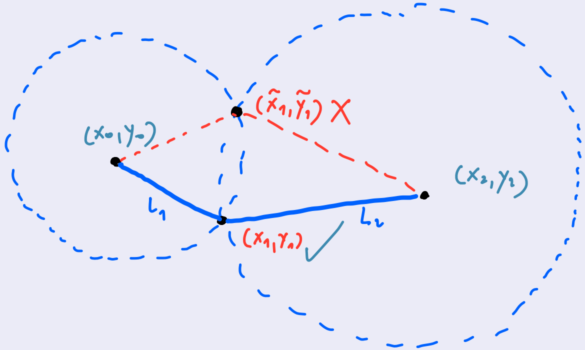
$$G(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}) = F(\boldsymbol{x}, \boldsymbol{y}) + \sum_{i=1}^{n+1} \lambda_i \left((x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 - L_i^2 \right).$$

- Potreben pogoj za minimum je

$$\begin{aligned}
&\nabla G(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}) \\
&= [\frac{\partial G}{\partial x_1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \frac{\partial G}{\partial x_2}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \ldots, \frac{\partial G}{\partial x_n}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \\
&\quad \frac{\partial G}{\partial y_1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \frac{\partial G}{\partial y_2}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \ldots, \frac{\partial G}{\partial y_n}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \\
&\quad \frac{\partial G}{\partial \lambda_1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \frac{\partial G}{\partial \lambda_2}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda}), \ldots, \frac{\partial G}{\partial \lambda_{n+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\lambda})] = \mathbf{0}.
\end{aligned}$$

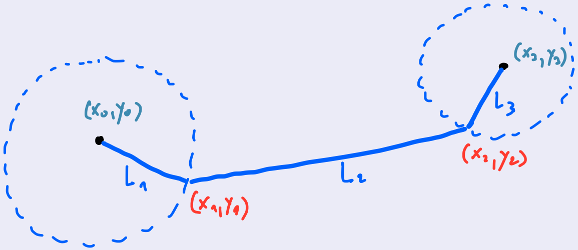
Primer ($n = 1$)

- *Dve palici.*
- *Geometrijska rešitev: presečišče dveh krožnic.*
- *Analitična rešitev: sistem dveh nelinearnih enačb z dvema neznankama.*



Primer ($n = 2$)

- *Tri palice.*
- *Geometrijska rešitev: ni jasna.*
- *Analitična rešitev: spet prevedemo na sistem DVEH nelinearnih enačb z dvema neznankama.*



$$X = (x_1, x_2)\ ;\ y = (y_1, y_2)\ ;\ \lambda = (\lambda_1, \lambda_2)$$

$$G(x, y, \lambda) = M_1 \frac{y_1 + y_0}{2} + M_2 \frac{y_2 + y_1}{2} + M_3 \frac{y_3 + y_2}{2}$$
$$+ \lambda_1 \left((x_1 - x_0)^2 + (y_1 - y_0)^2 - L_1^2 \right)$$
$$+ \lambda_2 \left((x_2 - x_1)^2 + (y_2 - y_1)^2 - L_2^2 \right)$$
$$+ \lambda_3 \left((x_3 - x_2)^2 + (y_3 - y_2)^2 - L_3^2 \right)$$

$$\frac{\partial G}{\partial x_1} = \cancel{2}\lambda_1 (x_1 - x_0) - \cancel{2}\lambda_2 (x_2 - x_1) = 0$$

$$\frac{\partial G}{\partial x_2} = \cancel{2}\lambda_2 (x_2 - x_1) - \cancel{2}\lambda_3 (x_3 - x_2) = 0$$

$$\frac{\partial G}{\partial y_1} = \frac{M_1}{\cancel{2}\,4} + \cancel{2}\lambda_1 (y_1 - y_0) + \frac{M_2}{\cancel{2}\,4} - \cancel{2}\lambda_2 (y_2 - y_1) = 0$$

$$\frac{\partial G}{\partial y_2} = \frac{M_2}{\cancel{2}\,4} + \cancel{2}\lambda_2 (y_2 - y_1) + \frac{M_3}{\cancel{2}\,4} - \cancel{2}\lambda_3 (y_3 - y_2) = 0$$

$$\frac{\partial G}{\partial \lambda_1} = (x_1 - x_0)^2 + (y_1 - y_0)^2 - L_1^2 = 0$$

$$\frac{\partial G}{\partial \lambda_2} = (x_2 - x_1)^2 + (y_2 - y_1)^2 - L_2^2 = 0$$

$$\frac{\partial G}{\partial \lambda_3} = (x_3 - x_2)^2 + (y_3 - y_2)^2 - L_3^2 = 0$$

Uvedemo: $\xi_1 = x_1 - x_0$, $\xi_2 = x_2 - x_1$, $\xi_3 = x_3 - x_2$, $\eta_1 = y_1 - y_0$, $\eta_2 = y_2 - y_1$, $\eta_3 = y_3 - y_2$

$$x_3 = x_0 + \sum_{j=1}^{3} \xi_j \,, \quad y_3 = y_0 + \sum_{j=1}^{3} \eta_j$$

Dobimo: $\lambda_1 \xi_1 - \lambda_2 \xi_2 = 0$, $\lambda_2 \xi_2 - \lambda_3 \xi_3 = 0$

$$\lambda_1 \eta_1 - \lambda_2 \eta_2 = -\frac{1}{2}\mu_1 \,, \quad \lambda_2 \eta_2 - \lambda_3 \eta_3 = -\frac{1}{2}\mu_2$$

$$\xi_1^2 + \eta_1^2 = L_1^2 \,, \quad \xi_2^2 + \eta_2^2 = L_2^2 \,, \quad \xi_3^2 + \eta_3^2 = L_3^2$$

$$\mu_1 = \frac{M_1 + M_2}{2} \,, \quad \mu_2 = \frac{M_2 + M_3}{2}$$

Sledi: $\lambda_1 \xi_1 = \lambda_2 \xi_2 = -\frac{1}{2\mu}$, μ neznanka $\Rightarrow$

$$-\frac{1}{2\mu}\frac{\eta_1}{\xi_1} + \frac{1}{2\mu}\frac{\eta_2}{\xi_2} = -\frac{1}{2}\mu_1 \,; \quad -\frac{1}{2\mu}\frac{\eta_2}{\xi_2} + \frac{1}{2\mu}\frac{\eta_3}{\xi_3} = -\frac{1}{2}\mu_2$$

Tako dobimo:

$$\frac{\eta_2}{\xi_2} = \frac{\eta_1}{\xi_1} - \mu\mu_1 \; ; \quad \frac{\eta_3}{\xi_3} = \frac{\eta_2}{\xi_2} - \mu\mu_2 = \frac{\eta_1}{\xi_1} - \mu\mu_1 - \mu\mu_2$$

$\frac{\eta_1}{\xi_1} = v$

$$= v - \mu(\mu_1 + \mu_2) \; ; \quad v \text{ mora neznanka}$$

Iz prejšnjih enačb sledi:

$$1 + v^2 = \frac{L_1^2}{\xi_1^2} \, , \quad 1 + (v - \mu\mu_1)^2 = \frac{L_2^2}{\xi_2^2} \, , \quad 1 + (v - \mu(\mu_1 + \mu_2))^2 = \frac{L_3^2}{\xi_3^2}$$

Torej: $\xi_1 = + \frac{L_1}{\sqrt{1 + v^2}}$; $\xi_2 = + \frac{L_2}{\sqrt{1 + (v - \mu\mu_1)^2}}$; $\xi_3 = + \frac{L_3}{\sqrt{1 + (v - \mu(\mu_1 + \mu_2))^2}}$

Neznanke η_i, $i=1,2,3$, se izražajo kot:

$$\eta_1 = \xi_1 v = \frac{L_1 \cdot v}{\sqrt{1+v^2}}; \quad \eta_2 = \xi_2 (v - \mu \mu_1) = \frac{L_2 (v - \mu \mu_1)}{\sqrt{1+(v-\mu\mu_1)^2}} \quad \text{in}$$

$$\eta_3 = \xi_3 \left(v - \mu(\mu_1 + \mu_2)\right) = \frac{L_3 \left(v - \mu(\mu_1+\mu_2)\right)}{\sqrt{1+\left(v-\mu(\mu_1+\mu_2)\right)^2}}$$

Sedaj iz zvez: $\sum_{j=1}^{3} \xi_j - (x_3 - x_0) = 0$ in $\sum_{j=1}^{3} \eta_j - (y_3 - y_0) = 0$,

dobimo dve enačbi za neznanki μ in v

Izrek

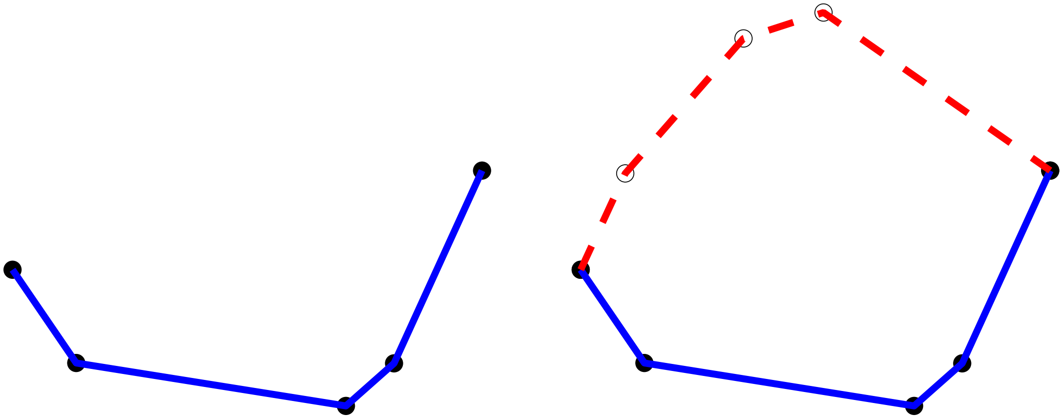
Če je $L_i < L_0$, $i = 1, 2, \ldots, n+1$, obstaja natanko ena verižnica, ki minimizira skalirano potencialno enegijo sistema palic. Določena je s stičišči

$$x_i = x_0 + \sum_{j=1}^{i} \xi_j(u,v),\ \ y_i = y_0 + \sum_{j=1}^{i} \eta_j(u,v), \quad i = 1, 2, \ldots, n.$$

Pri tem za $i = 1, 2, \ldots, n+1$ velja

$$\xi_i(u,v) = \frac{L_i}{\sqrt{1 + \left(v - u \sum_{j=1}^{i-1} \mu_j\right)^2}},\ \ \eta_i(u,v) = \xi_i(u,v)\left(v - u \sum_{j=1}^{i-1} \mu_j\right),$$

kjer je $\mu_i = (M_i + M_{i+1})/2$, $i = 1, 2, \ldots, n$, in sta u ter v rešitev sistema $U(u,v) = \sum_{i=1}^{n+1} \xi_i(u,v) - (x_{n+1} - x_0) = 0$, $V(u,v) = \sum_{i=1}^{n+1} \eta_i(u,v) - (y_{n+1} - y_0) = 0$.

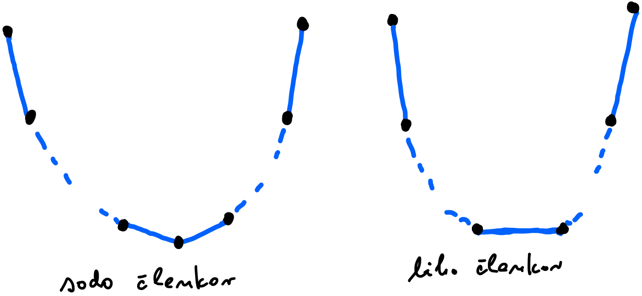


Simetrične verižnice

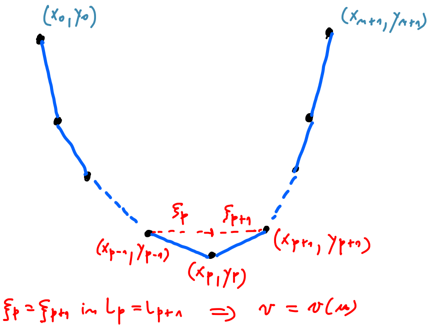
- Denimo, da so členki verižnice simetrični, torej

$$L_i = L_{n+1-i+1},\ M_i = M_{n+1-i+1}, \quad i = 1, 2, \ldots, \lfloor (n+1)/2 \rfloor,$$

in $y_{n+1} = y_0$.

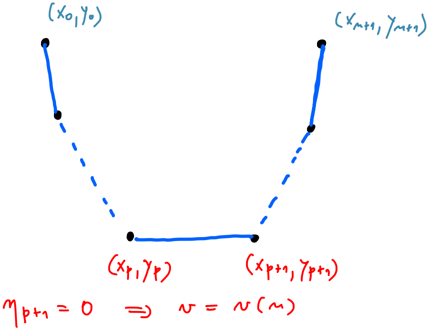


Sodo mnogo členkov: $n + 1 = 2p$



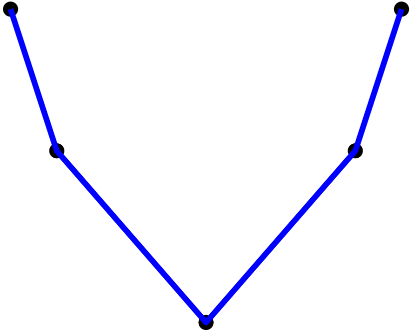
- Srednja členka sta enako dolga, horizontali med krajiščema srednjih palic sta enaki.
- Neznanka v se izrazi z neznanko u in dobimo eno enačbo z eno neznanko.

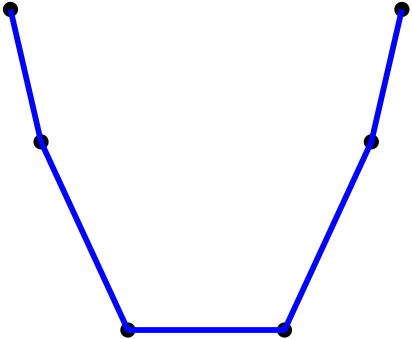
Liho mnogo členkov: $n + 1 = 2p + 1$



- Srednji členek je vodoraven.
- Neznanka v se izrazi z neznanko u in dobimo eno enačbo z eno neznanko.

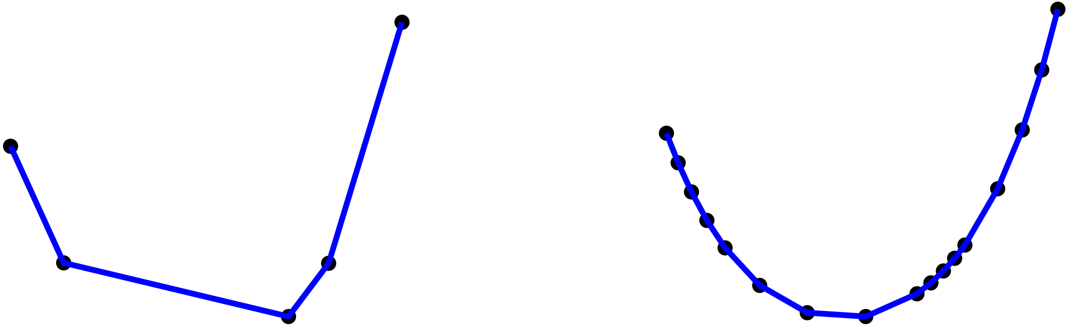
- Simetrična verižnica s sodo členki $L_1 = 2$, $L_2 = 3$, $L_3 = 3$ in $L_4 = 2$ (levo).
- Simetrična verižnica z liho členki $L_1 = 2$, $L_2 = 3$, $L_3 = 2$, $L_4 = 3$ in $L_5 = 2$ (desno).





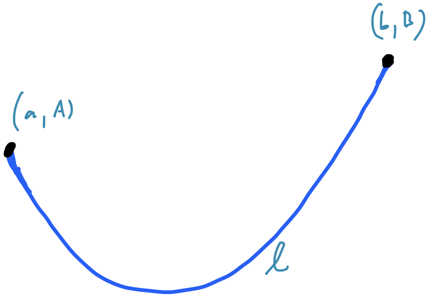
Limitni primer: $n \to \infty$

- Kaj se dogaja, ko število členkov raste, vsota dolžin pa je konstantna?
- Diskretna verižnica se približuje povešeni vrvi.
- Kako določiti njeno obliko?



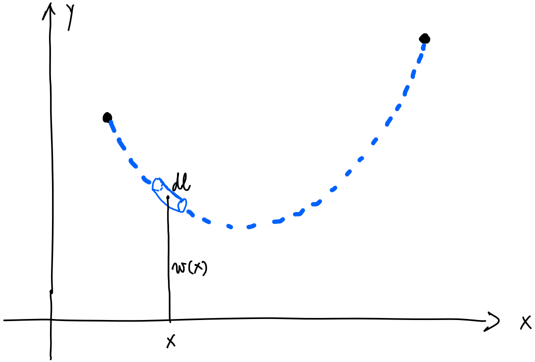
Zvezna verižnica

- Gibka homogena vrv dolžine ℓ.
- Konca sta pripeta v $T_1 = (a, A)$ in $T_2 = (b, B)$.



Matematični opis

- Vrv ponovno "želi" minimizirati potencialno energijo.
- Denimo, da je oblika vrvi podana s funkcijo $w : [a, b] \to \mathbb{R}$.
- Izpeljemo lahko izraz za potencialno energijo delčka vrvi dolžine $d\ell$.



Potencialna energija delčka vrvi dl je:

$$dW = g\, w(x)\, dm = g\, w(x)\, \rho\, dV$$

$$= g\, w(x)\, \rho \cdot S\, dl$$

g - težni pospešek, ρ - gostota vrvi; S - prečni presek

Potencialno energijo lahko skaliramo z $g \rho S$ in upoštevamo $dl = \sqrt{1+w'(x)^2}\, dx$:

$$F(w) = \int_a^b w(x) \sqrt{1+w'(x)^2}\, dx.$$

Minimiziramo torej

$$\min_{w \in \mathcal{C}^1([a,b])} F(w), \quad \text{pri pogoju } l = \int_a^b \sqrt{1+w'(x)^2}\, dx.$$

- Iščemo torej

$$\min_{w\in\mathcal{C}^1([a,b])} F(w), \quad \text{pri pogoju } \ell = \int_a^b \sqrt{1+w'(x)^2}dx,$$

kjer je

$$F(w) = \int_a^b w(x)\sqrt{1+w'(x)^2}dx.$$

- Klasični problem variacijskega računa.
- Rešitev je bila znana že 1691 (Gottfried Wilhelm Leibniz, Christiaan Huygens, Johann Bernoulli).
- Oblika ni parabola, kar je zaznal že Galileo Galilei, dokazal pa Joachim Jungius leta 1669 (posthumna objava).

Eksplicitna rešitev

- Eksplicitna rešitev se izraža kot

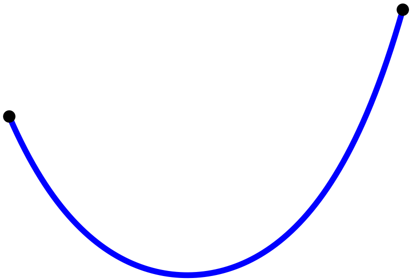
$$w(x) = \lambda + C\cosh\left(\frac{D-x}{C}\right),$$

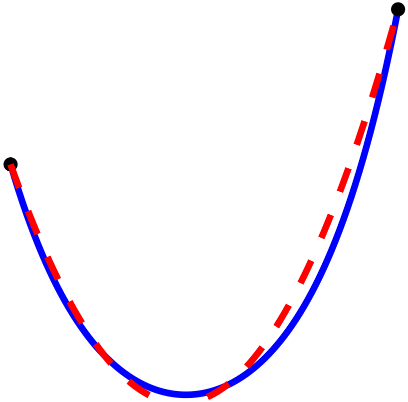
kjer konstante λ, C in D določimo iz pogojev

$$w(a) = A, \quad w(b) = B, \quad \int_a^b \sqrt{1+w'(x)^2}dx = \ell.$$

Primer: $T_1 = (0, 1)$, $T_2 = (5, 3)$, $\ell = 10$

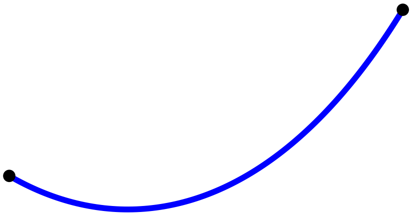
- Modra je zvezna verižnica, rdeča je parabola.

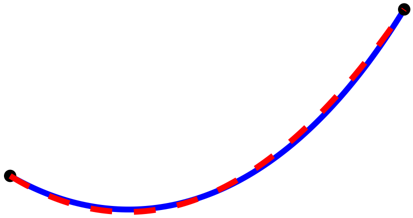




Primer: $T_1 = (0, 1)$, $T_2 = (5, 3)$, $\ell = 6$

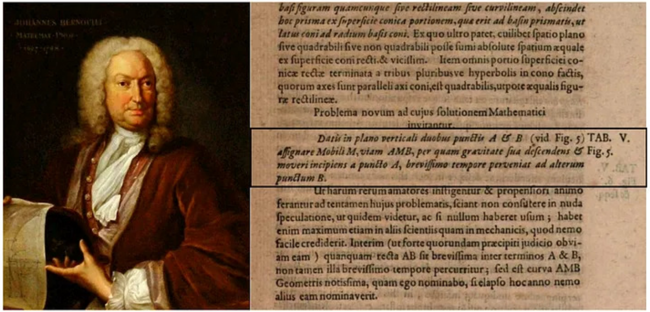
- Modra je zvezna verižnica, rdeča je parabola.





Brahistohrona

- Še ena od klasičnih ravninskih krivulj.
- Kroglico spustimo z višje točke proti nižji, da drsi po krivulji brez trenja.
- Po kateri krivulji je najhitreje v nižji točki?
- Problem je leta 1696 zastavil Johann Bernoulli.



baſi figuram quamcunque ſive rectilineam ſive curvilineam, abſcindet hoc priſma ex ſuperficie conica portionem, quæ erit ad baſin priſmatis, ut latus coni ad radium baſis coni. Ex quo ultro patet, cuilibet ſpatio plano ſive quadrabili ſive non quadrabili poſſe ſumi abſolute ſpatium æquale ex ſuperficie coni recti, & viciſſim. Item omnis portio ſuperficiei conicæ rectæ terminata a tribus pluribusve hyperbolis in cono factis, quorum axes ſunt paralleli axi coni, eſt quadrabilis, utpote æqualis figuræ rectilineæ.

Problema novum ad cujus ſolutionem Mathematici invitantur.

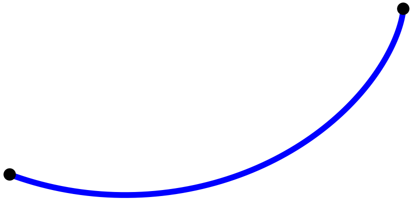
Datis in plano verticali duobus punctis A & B (vid. Fig. 5) TAB. V. *aſſignare Mobili M, viam AMB, per quam gravitate ſua deſcendens &* Fig. 5. *moveri incipiens a puncto A, breviſſimo tempore perveniat ad alterum punctum B.*

Ut harum rerum amatores inſtigentur & propenſiori animo ferantur ad tentamen hujus problematis, ſciant non conſiſtere in nuda ſpeculatione, ut quidem videtur, ac ſi nullum haberet uſum; habet enim maximum etiam in aliis ſcientiis quam in mechanicis, quod nemo facile crediderit. Interim (ut forte quorundam præcipiti judicio obviam eam) quanquam recta AB ſit breviſſima inter terminos A & B, non tamen illa breviſſimo tempore percurritur; ſed eſt curva AMB Geometris notiſſima, quam ego nominabo, ſi elapſo hoc anno nemo alius eam nominaverit.

Slika: Johann Bernoulli (levo) in originalni opis problema brahistohrone v Acta Eruditorum. (Vir: https://www.cantorsparadise.com)

- Rešili so ga Isaac Newton (“Tanquam ex ungue leonem!”), Jacob Bernoulli, Gottfried Leibniz, Ehrenfried Walther von Tschirnhaus in Guillaume de l’Hôpital .
- Rešitev je krivulja, ki je odsek cikloide.

Brahistohrona med točkama $T_1 = (5,3)$ in $T_2 = (0,1)$



- Brahistohrona je v parametrični obliki dana kot

$$x = x_1 + \frac{1}{2}k^2(t - \sin t),$$
$$y = y_1 - \frac{1}{2}k^2(1 - \cos t),$$

$t \in [0, t_{\max}]$.

References

- E. Zakrajšek, *Matematično modeliranje*, DMFA-založništvo, Ljubljana, 2004.
- K. Veselić, *Finite catenary and the method of Lagrange*, SIAM Rev. 37 (1995), no. 2, 224–229.