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Left time derivatives in mathematics, mechanics and control of motion

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ABSTRACT

By the traditional representation accepted in mathematics, mechanics and theoretical physics, the time-derivatives are defined as the *right* derivatives and are used in this way in differential equations describing processes in nature and technology. The fact that even for infinitely smooth $x(t)$, the right time-derivatives do *not* physically exist, due to the *positive* orientation of time, somehow escaped the attention of scientists. This led to misconceptions and omissions in mechanics, physics and engineering, with unexpected consequences in some cases. All measurements and experiments contain and use only *left* time-derivatives, thereby with time delays. All processes require some kind of transmittal of information (forces, actions) which takes time, so the expressions that define their evolution from a current state actually contain the left and delayed time derivatives, even if they are written with the exact right time-derivatives, according to the classical tradition. In this paper, the causal representations of physical processes by differential equations with the *left* time-derivatives on the right-hand side are considered for some basic problems in classical mechanics, physics and technology. The use of the left time-derivatives explicitly takes into account the causality of processes depending on the transmission of information and defines the motions subject to external forces that may depend on accelerations and higher order derivatives of velocities. Such forces are exhibited in Weber's electro-dynamic law of attraction; they are produced by the Kirchhoff–Thomson adjoint fluid acceleration resistance acting on a body moving in a fluid, and they are also involved in the manual control of aircraft or spacecraft that depends on accelerations of the craft itself. The consistency condition is presented, and the existence of solutions for equations of motion driven by forces with higher order derivatives of velocity is proved. The inclusion of such forces in the autopilot design is proposed to assure the safety of the aircraft in case of a failure of its outboard velocity sensors. It is demonstrated that the classical form of the 2nd law of Newton is preserved with respect to the *effective* forces for which the parallelogram law of addition is valid. Then the Lagrange and Hamilton equations are extended to include the generalized forces with the left higher order derivatives, and a method for the solution of such equations with the left and delayed higher order derivatives is presented with the example of a physical pendulum. The results open new avenues in science and technology providing the basis for correct design in the projects sensitive to information transmittal.

1. Introduction

By an old tradition, the notion of the derivative at a point z is defined as the *right* limit: $f'(z) = df/dz = \lim [f(z + \Delta z) - f(z)]/\Delta z$ as $\Delta z \to 0$, $\Delta z > 0$. If $f(z)$ is a physical process and z denotes a moment of time, $z = t^*$, then $f'(t^*)$ is *non-causal* since $t^* + \Delta t > t^*$ is physically inexistent, although in the abstract sense, when $f(t)$ is supposed to be known for all the future $[t^*, \infty)$, it is mathematically correct, and rigorous theories in differential equations have been developed

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to describe the evolution of processes $f(t)$ under tacit assumption that those processes, indeed, follow certain future curve $f(t)$ obtained from a differential equation postulated for the time interval $t \in [t^*, t^* + \Delta t)$ and extended to the future in its *original setting* for the entire half-axis $[t^*, \infty)$.

Real processes in nature and technology may, however, contain some inputs (controls) from other processes (forces) that may influence the original process under consideration by providing some signals (actions) being measured or directly transmitted onto the original process, without following the tacit assumption for the rigid behavior of the original process $f(t),\ t \in [t^*, t^* + \Delta t)$. For this reason, the *right* derivative $f'(t)$ written above is known only over the registered past history of the process, and it is *physically inexistent* at the current moment t^* since the values $f(t^* + \Delta t)$, $\Delta t > 0$, are not yet realized, thus, unknown. It means that feedback controls and other inputs on the right-hand side of equations of motion that depend on the higher order *right* derivatives are in contradiction with the physical reality, and for that reason they are formally excluded in major works on mechanics [1–5] and in textbooks that are followed in engineering projects.

To remove this heavy restriction on the control of motion and processes imposed by the old tradition, the *left* and *delayed* time-derivatives are considered in the control of motion and in the process equations to assure the proper functioning and stability, and to improve the vital control systems, such as the autopilot in aviation and flight control systems in spacecrafts, by the acceleration assisted control with the onboard accelerometers.

New representations of some physical laws are considered in this paper, and causality of differential systems is studied in relation to the orientation of time. Then, geometry and the time phenomena in classical mechanics are revisited, and the new forms of generalized equations of motion are derived with *left* and *delayed higher order* time-derivatives on the right-hand side. The method for their integration is demonstrated by example of a physical pendulum, and an application to the computerized autopilot design is proposed, with the acceleration assisted control to assure the safety of the aircraft in case of a failure of its outboard velocity sensors (*Pitot tubes*).

The paper is organized as follows. In Section 2, representations of Newton's second law of motion are considered with the generalization of this law for bodies with variable masses due to Buquoy [6] and later Mestschersky [7] and Levi-Civita [8]. In Section 3, the problems related to time orientation and causality are discussed, and in Section 4, the forces with left and delayed higher order derivatives of velocity are introduced into the general equation of motion. In Section 5, some basic equations of theoretical physics are listed where the consideration of the left and delayed time-derivatives is necessary, with the basic lemma that allows us to reduce the equations with the left time-derivatives to the usual equations currently considered in mathematics. In Section 6, the existence of solutions is proved under certain consistency condition related to the left highest order time-derivative on the right-hand side. In Section 7, the fields of *effective* forces are considered, and the parallelogram law is verified for effective forces in linear systems depending on accelerations on right-hand sides. Section 8 presents an application to the autopilot design in aviation, to expose the necessity of acceleration assisted control. Section 9 presents an application to the reactive motion of a spacecraft with variable mass and acceleration assisted control. In Section 10, the equations in independent coordinates for motion of bodies with variable masses and left higher order derivatives are studied, and the Lagrange and Hamilton equations are presented that include generalized forces with left higher order derivatives. Section 11 presents a method of integration for the equations with left and delayed higher order time-derivatives on the right-hand side on the example of a physical pendulum. In Section 12, some points of interest are summarized, followed by references immediately relative to the problems considered.

2. Representations of Newton's second law of motion

The second law of Newton states: "Law II. The change of motion is proportional to the motive force impressed and is made in the direction of the right line in which that force is impressed" [1], see also [5, p. 259]. In high school textbooks, this law is written in the form: $ma = F$, where m is a constant mass, a the acceleration, and F is "the motive force impressed" or simply "a force", a self-explanatory notion known from life experience. In university textbooks, it is specified in more exact terms:

$$mx'' = F(t, x(t), v(t)), \qquad v(t) = x'(t), \qquad x''(t) = v'(t) = a(t), \qquad x(0) = x_0, \qquad v(0) = v_0, \quad t \geq 0, \tag{1}$$

which define a particular motion starting at x_0, v_0, with velocity $v(t)$ defined as time derivative

$$v(t) = x'(t) = dx/dt = \lim\ [x(t + \Delta t) - x(t)]/\Delta t \quad \text{as } \Delta t \to 0, \Delta t > 0. \tag{2}$$

Widely used representations (1)–(2) impose heavy restrictions in mechanics and control theory which restrictions are not necessary and can be removed.

The first generalization of Newton's second law for reactive forces was proposed 200 years ago by Buquoy [6]. When $m = const$, the first formula in (1) can be written as follows:

$$mx'' = mv'(t) = m dv/dt = d(mv)/dt = F(t, x(t), v(t)), \quad t \geq 0. \tag{3}$$

If $m \neq const$, then the last equality in (3) presents a more general form of Law II:

$$d(mv) = m\, dv + v\, dm = F(t, x(t), v(t))dt, \quad t \geq 0,\ dt > 0, \tag{4}$$

where differentials can be viewed as small increments, this leading to the well known interpretation: "the change of momentum, $d(mv)$, equals the impulse of force (or simply impulse), Fdt". If $m = const$, then $dm = 0$, and (4) coincides with (1) as $dt \to 0$. If mass $m = m(t) \neq const$, then (4) accounts for the changing mass of a moving body when dm, moving with the *same* velocity $v(t)$, separates from the body. However, if elementary mass dm is *ejected* from the body (e.g., as burnt fuel) with a different velocity $w(t) \neq v(t)$, it will impress an additional force upon the body which force must be proportional to the additional "change of motion" (see Newton's second law cited above), i.e., to the additional change of momentum which is itself proportional to the relative velocity $v - w$ with which dm is ejected from the body. Thus, the quantity vdm shown in (4) should be replaced by the quantity $(v - w)dm$, yielding the equation

$$mdv + (v - w)dm = F(t, x(t), v(t))dt, \quad t \in [0, T), \tag{5}$$

where velocities v and w are absolute velocities of the body and the ejected mass dm respectively, in a coordinate frame at rest in which the motion of a body is considered. The reader can see the change in the force impressed on the body by the mass dm being ejected, if (5) is rewritten in the form which corresponds to the form in (1), (3):

$$mdv = F(t, x(t), v(t))dt + (w - v)dm, \quad t \in [0, T). \tag{6}$$

Here the change in momentum of a body is on the left, and all impulses are on the right of the equation. Now, if m is constant ($dm = 0$) or is being separated from the body without ejection ($w = v$, $dm < 0$), then the force is *not* changing, only the mass $m(t)$ of the body is decreasing and acceleration increasing since the same force is acting on decreasing mass of the body. In this case, the last term on the right is zero, and (6) coincides with (1). However, in a spacecraft with a jet engine, the burnt fuel mass is ejected, $dm < 0$, with velocity w different from the velocity v of the spacecraft. To explain the action of ejected mass dm in (6), we assume, for simplicity, that w, v are collinear vectors. If the burnt fuel mass is ejected in the same direction in which the spacecraft moves, so that $w > v$, then additional reactive force exerts the braking effect upon the spacecraft since $(w - v)dm < 0$. If it is ejected in the opposite direction, so that $w - v < 0$, then additional reactive force accelerates the motion since $(w - v)dm > 0$. However, the entries in (1)–(6) can be considered as $3D$ vectors (except time t, dt and mass m, dm which are scalars), so that turning the funnel ejecting the burnt fuel mass allows one to control also the direction of the motion. The term $(w - v)dm$ added to the nominal impulse $F(.)dt$ in (6) represents, in fact, the control impulse $u(t)dt$ in the resulting total impulse $F^*(.)dt = [F(.) + u(t)]dt$, yielding the equation of controlled motion:

$$m(t)\, dv/dt = F^*(.) = F(t, x(t), v(t)) + u(t), \qquad u(t) = [w(t) - v(t)]dm/dt, \quad t \in [0, T). \tag{7}$$

Burnt fuel generates not only the reactive force of ejected masses but also a direct active force of heated gas pressure which is considered a part of $F(.)$ in (5)–(7), but can be studied as separate action, see Section 9. It is worth noting that Eq. (7), quite different from (1)–(4), can be included in the original Newton's statement of Law II above since it is not specified what "the motive force impressed" actually is. This emphasizes the importance of particular *symbolic* representations.

Eqs. (5), (6) represent a fundamental generalization of the classical equations of motion (1)–(4), very important for applications (as we know today). However, when published in 1815, see [6], this generalization was not properly recognized, not entered in textbooks, and thus, quickly forgotten. So, it was rediscovered by Mestschersky in 1897, see [7], where many special cases are also studied. Then in 1928, the equation $d(mv)/dt = F$, cf. the right equality in (3), was independently derived by Levi-Civita [8], representing the case $w = 0$, that corresponds to the motion of a body with variable mass being ejected with relative velocity $w^* = w - v = -v$, thus, excluding the control of motion by means of ejected burnt fuel mass.

3. Time orientation and causality

In the literature, velocity $v(t)$ on which the motive force $F(.)$ in (1) may depend is defined as the right derivative through the limit in (2). However, at the moment, t of actual motion, the value $v(t + \Delta t)$ does *not* exist for any $\Delta t > 0$. This means that the limit in (2) also does not exist, so that Eq. (1) refers, in fact, to some prospective values of $v(t)$ in future, thus being *non-causal*. The reader may object: well, then what is shown on the speedometer of a car? Yes, the velocity is shown which is actually measured as *left* time-derivative $v(t) = \lim\, [v(t) - v(t - \Delta t)]/\Delta t$, $\Delta t \to 0$, $\Delta t > 0$, not right derivative as written in (2). This reflects the positive orientation of time: suppose that $x(t)$ in (1) is a distance of the moving mass m from the origin if the motion has started at time $t = 0$ with initial conditions indicated in (1). If we consider a moment $t^* > 0$ with the past history of motion *registered* in a measuring device or in a computer over the segment $[0, t^*]$, then over the interval $(0, t^*)$ there exist both right and left derivatives; at the moment $t = 0$, there exists only right derivative; at $t = t^*$ there exists only left derivative, and over the future interval (t^*, T), $T \leq \infty$, there is no motion yet, thus no derivatives exist, and the same on the interval $(-\infty, 0)$ when there was no motion at all. This concerns all natural processes (physical, biological, etc.) developing in time: right time-derivatives may exist only in the *registered past history* of a process. Of course, right derivatives at the current moment, as well as future situations and/or decisions (called rational expectations), can be postulated (imagined as desired) and taken into account, which is routinely done in economy and finance; but in engineering and technology, it may be improper and needless to do so. In natural sciences, there is another way to include current accelerations and other higher order time-derivatives into process equations, thereby retaining their causality.

In control of motion, the effect of time orientation is compounded by time uncertainty. Indeed, velocity $v(t)$ as left derivative continuously measured by the speedometer in a car appears on the driver's panel with a delay $\delta > 0$ due to the finite speed of information transmittal. Hence, at the moment $t = t^*$, a driver sees the velocity $v(t^* - \delta)$, not the actual velocity $v(t^*)$. However, in Eq. (1) of the motion, the force $F(.)$ is *impressed* (not measured by a device, but *felt* as are, e.g., gravitational or resistance forces), thus, at a moment t^*, we have the force $F(t^*, x(t^*), v(t^*))$ acting without delay if there is no information transmittal for the values $x(t)$, $v(t)$, in which case time-uncertainty is not implicated in the motion governed by the laws of mechanics such as Law II above. In contrast, if the control $u(.)$ in (7) depends on certain parameters which are *measured* on the trajectory and transmitted into the power train of the motion, then $u(t-\delta)$ actually depends on $\delta > 0$, at each moment $t > 0$, through those measured parameters.

4. Forces with left and delayed higher order time-derivatives

Consider the specification of Newton's Law II presented by Eq. (1) which can be found in all books on mechanics and related subjects. A distinctive feature of this equation is that "the motive force impressed" $F(.)$ is defined for the moment t and depends only on t and/or $x(t)$ and/or $v(t)$. In some textbooks, it is explained that force $F(.)$ does not depend on acceleration $a(t) = x''(t)$, because if it did, we would have the equation $mx'' = F(t, x, v, x'')$ which, if solved for x'', would render $x'' = F^*(t, x, v, m)$, hence, the right-hand side $F(.)$ would *not* be "the motive force impressed" in the sense of Newton's Law II, but rather it would be $F^*(t, x, v, m)$ which does not depend on $x''(t)$ again. What would happen if $F(.) = F(t, x, v, x'', x''')$ is not even mentioned since such a consideration is taken as an obvious blunder.

However, equations of motion with variable mass contain controls: w in (5)–(6), or $u(t)$ in (7), and it is not clear why w and $u(t)$ must not depend on acceleration $x''(t)$ and its rate of change $x'''(t)$. In fact, they can, and the so called acceleration assisted control is widely used in practice for soft regulation, despite its contradiction with (1)–(4). Indeed, consider the following railway construction principle. If to change the direction of motion, a perfect circular arc is joined to a right line segment of a railway, then at the connection point the train will receive a hard impact of centripetal force, and the train may derail if its speed is high enough. If a person is standing on the platform of a coach with a door open, he will be thrown out of the train by centrifugal force. To avoid such eventualities, the railway connection must be designed as a cubic or higher degree curve in order to soften the turn and eliminate hard impacts by means of a correct profile of the railway. Obviously, the same concerns the profile of a highway. Whatever the actual profile of a road, experienced drivers always soften a turn by crossing the lanes while continuously turning the steering wheel (this cannot be done by a train because of the rails on which it runs). With manual control, the pilot of an aircraft or spacecraft does the same by making a turn along some higher degree curve following his personal feeling of the centrifugal force that appears during the turn. As a matter of fact, in all *manually* controlled vehicles, a turn is being done by a control $u(.)$ which is called in theory, "open loop control $u(t)$", being, in reality, a feedback control $u(t, x(t), x''(t), x'''(t))$ depending on actual acceleration $x''(t)$ and its rate of change $x'''(t)$ felt by the pilot, and, maybe, on higher order derivatives if they are felt by a human being (an open question for medicine). In manually controlled aircraft, the pilot always employs a feedback control of the form $u(t-\delta, x(t-\delta),\ x''(t-\delta),\ x'''(t-\delta))$ which depends on time t (with delays $\delta > 0$ due to a finite speed of information transmittal in human senses) and distance $x(t)$, if it is seen during landing, but does not depend on v since constant velocity is *not* felt by a human being nor by instruments on board, according to the postulate of physical equivalence of all inertial systems [3]. This is the celebrated and universally recognized principle of relativity used with great success by Einstein [9]. Dependence on velocity $v(t)$ means, in fact, dependence on acceleration dv/dt which accompanies a *varying* velocity $v(t-\delta)$. A manual control $u(.)$ always depends on the acceleration $x''(t-\delta)$ and its rate of change $x'''(t-\delta)$, no matter that they are theoretically excluded by a choice of representation in the equations of motion (1), (7). Therefore, it is important to extend the real life situation in manual control onto the *automatic* control systems by removing the existing restriction with a new choice of representation for Newton's Law II, which would allow left and delayed higher order time derivatives on the right-hand side of (1)–(7), i.e., the forces dependent on such derivatives.

Consider Eq. (7), where w and/or dm/dt, thus $u(t)$, may depend on acceleration and higher order derivatives. Dividing (7) by $m(t) > 0$ and using the *left* time-derivatives on the right-hand side for $t > 0$, we can write the *causal* representation of the general equation of motion in the form:

$$x'' = dv/dt = [F(t, x(t), v(t)) + u(t)]/m(t) = F^*(t, x, x'^{-}, x''^{-}, \dots, x^{(k)-}), \qquad x(0) = x_0, \qquad x'^{-}(0) = v_0, \tag{8}$$

where the superscript $(^-)$ indicates the left time-derivative of the corresponding variable which is written in normal script for better visibility. The only right time-derivative is $x'' = dv/dt$, on the left in (8) due to the forward propagation of motion. It is clear that $F^*(.)$ on the right in (8) is well defined for all $t > 0$, where for simplicity, the time-uncertainty $\delta > 0$ in $(t-\delta)$ is not shown, as well as $m(t)$ which is not shown explicitly as a variable of $F^*(.)$ in Eq. (8).

5. Left derivatives in theoretical physics

Transmission of information takes time, which brings about uncertain time delays. This and the left time-derivatives in the intertwined physical or chemical processes, or under some kind of control or disturbances, may affect the natural physical process or motion under consideration [10, Sec. 3,4]. In a moving system observed from another still or moving system, the relativistic effects take place supplying an *image* [10, Sec. 8–10,13] of the observed process, and in that image, the time-derivatives which represent forces or conditions acting upon the process should be considered as delayed left time-derivatives.

Forces containing higher order derivatives can appear in equations of motion not only through controls. Such forces depending on accelerations have been considered by Sir Horace Lamb in equations of motion of a solid in ideal liquid, see [11, p. 168, Section 124, Equations (1)] with reference to Kirchhoff and Sir W. Thomson (1871), where forces of the fluid pressure linearly depended on the acceleration of the solid itself, see [11, p. 168, Equations (2); p. 169, Equations (3)]. Such forces usually can be taken into account by the introduction of adjoint masses, see the example given in [11, p. 190, Section 137, Equations (2)] with reference to Thomson and Tait [12, Art. 321]. The author thanks V.V. Rumyantsev for these references.

The natural phenomenon with resistance in $F^*(.)$ depending on the acceleration of a solid falling into a viscous liquid is presented in [13, p. 181] and [14, p. 34]. For an application of acceleration assisted hovercraft control, see [13, pp. 179–180] and [14, pp. 39–41]. Of course, the accelerations in such forces are represented by the *left* time-derivatives.

Another example is furnished by Weber's electro-dynamic law of attraction, the force per unit mass being $F^* = [1 - (r'^2 - 2rr'')/c^2]/r^2$, where r is the distance of a particle from the center of force (W. Weber, Ann. der Phys. LXXIII, 1848, p. 193), see also [2, p. 45]. In this force, the values r', r'' are also given by the left time-derivatives.

In the well known equation of the elementary electric oscillating circuit containing a coil of inductance L, a resistance R and capacitance C: $Lq'' + Rq' + q/C = 0$, $q' = dq/dt$, where $q(t)$ is the charge on the capacitor, the first term contains the *right* time-derivative of the second order propelling the oscillations in the immediate future time moment, but the term $Rq'(t)$ acting on the process (resistance voltage) is given by left time-derivatives.

The similar situation, though a little more involved, can be seen in partial differential equations, some of which we reproduce from [15, pp. 385–389, 605–618].

The Continuity Equation (the law of conservation of mass in fluid mechanics):

$$d\rho/dt + \rho \operatorname{div} v = 0 \quad \text{or} \quad \partial\rho/\partial t + \rho \operatorname{div} v + v\nabla\rho = 0, \tag{9}$$

$\operatorname{div} v = \partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z$ (divergence of v), $\nabla\rho = (\partial\rho/\partial x, \partial\rho/\partial y, \partial\rho/\partial z)$, where $\rho(x, y, z, t)$ and $v(x, y, z, t)$ are density and vector velocity of the fluid.

Eulerian equation of motion of an ideal fluid:

$$dv/dt = F - \nabla p/\rho, \tag{10}$$

where F is the intensity of the mass force field and $p(x, y, z, t)$ is the pressure.

Navier–Stokes' equation of motion of an incompressible fluid ($\operatorname{div} v = 0$):

$$dv/dt = F - \nabla p/\rho + \mu\Delta^2 v, \tag{11}$$

where $\mu = \eta/\rho$ is the *kinematic viscosity* of the fluid and Δ^2 is the Laplacian operator.

Maxwell's equations for macroscopic (averaged by time intervals charges and currents concentrated in finite volumes of separate particles) electromagnetic fields in SI units:

$$\operatorname{curl} E = -\partial B/\partial t, \qquad \operatorname{div} B = 0, \qquad \operatorname{curl} H = j + \partial D/\partial t, \qquad \operatorname{div} D = \rho, \tag{12}$$

where the vector E is the intensity of the electric field, B is the magnetic induction, H is the intensity vector of the magnetic field, j is the current density vector, D is the electric displacement vector, and ρ is the volume density of free charges: $\rho = dq/dV$.

At each point in space there are micro-fields: an electric one of intensity e and a magnetic one of intensity h, which satisfy the *Lorentz equations*, in SI units:

$$\operatorname{curl} e = -\mu_0\partial h/\partial t, \qquad \operatorname{div} e = \rho/\epsilon_0, \qquad \operatorname{curl} h = j + \varepsilon_0\partial e/\partial t, \qquad \operatorname{div} h = 0. \tag{13}$$

Here ε_0 is the permittivity of free space, μ_0 is the magnetic constant (permeability of free space), and $j = \rho v$, where v is the velocity of the charges.

In Eqs. (9)–(13), the ordinary or partial time-derivatives are right derivatives that propel the propagation of fields, and other derivatives are taken with respect to space coordinates of the propagating field which are changing with time and represent the point-wise conditions (actions). In causal consideration, these derivatives must be *left* derivatives since they relate to the past, already occurred, state of the field that influences the field evolution in time.

The causal equation (8) with the left higher order time-derivatives can be solved by the standard methods of ordinary differential equations, for which we need the following lemma.

Lemma 5.1 (*[13,14]*)**.** *If a function $x(t)$ is defined on an open interval (a, b) and has continuous left derivative on (a, b), then $x(t)$ is continuously differentiable on (a, b).*

Proof. By hypothesis, for every $t \in (a, b)$, there is a limit

$$x'^{-}(t) = \lim_{\Delta t \to +0} [x(t) - x(t - \Delta t)]/\Delta t, \quad t - \Delta t \in (a, b), \tag{14}$$

which, as a function of t, is continuous on (a, b), that is

$$\lim_{t \to t_0} x'^{-}(t) = x'^{-}(t_0), \quad t_0 \in (a, b). \tag{15}$$

Let $t - \Delta t = t_0$, then (14) can be rewritten as follows, yielding the right derivative at t_0:

$$\lim_{\Delta t \to +0} [x(t_0 + \Delta t) - x(t_0)]/\Delta t = x'^{+}(t_0) \equiv x'(t_0), \quad t_0 + \Delta t = t \in (a, b). \tag{16}$$

Since by construction,

$$[x(t) - x(t - \Delta t)]/\Delta t \equiv [x(t_0 + \Delta t) - x(t_0)]/\Delta t, \quad \forall t \in (a, b),\ \forall t_0 = t - \Delta t \in (a, b), \tag{17}$$

so, from (14), (16), (17), we have $x'^{-}(t) = x'^{+}(t_0) \equiv x'(t_0)$, which, due to (15), implies

$$x'^{-}(t_0) = x'^{+}(t_0) \equiv x'(t_0), \tag{18}$$

as $\Delta t \to +0,\ t \to t_0$ for every $t_0 \in (a, b)$. □

Remark 5.1. Left and right derivatives considered above are special cases of Dini derivatives and the Lemma, in a more general setting, corresponds to the Denjoy–Young–Saks Theorem [16], where only finiteness of a one-sided derivative is required for every $t \in (a, b)$, implying differentiability of $x(t)$ almost everywhere in (a, b).

Remark 5.2. As follows from (17) with $t = (t_0 + \Delta t) \to t_0 + 0$, as $\Delta t \to +0$, left derivatives in (8) can be regarded as delayed right derivatives: $x^{(k)-}(t) \equiv x^{(k)+}(t_0) = x^{(k)+}(t - \Delta t)$, as $\Delta t > 0$. This, however, leads to theoretical complications and may result in the loss of stability which might not be the case for the original equation (8), see Section 9. For these reasons, we do not use such representations.

Remark 5.3. It is clear that Lemma 5.1 is valid not just for functions of time $x(t)$ but for any functions $f(x)$, and not only for ordinary derivatives in Eqs. (9)–(11) of theoretical physics but for partial derivatives as well. It explains why the consideration of some *right* derivatives, though non-causal and physically inexistent, renders *correct final* results in real life processes and experiments. However, the use of *all right* derivatives in the equations of theoretical physics leads to a handicap of non-consideration of the *left higher order* derivatives which may essentially affect the process, see Section 9, though not being the main forcing derivatives on the left of process equations in the normal form of Cauchy, as in the 2nd law of Newton in (1). In this sense, Eq. (8) presents a (non-relativistic) generalization of the 2nd law of Newton to the case with left higher order derivatives on the right-hand sides of an equation of motion which really exist in the acceleration assisted control and essentially affect the motion as shown in Sections 6–9, 11. Clearly, this situation relates not only to mechanics, but to all physical, chemical and other processes as well.

6. Consistency condition and existence of solutions

The continuity of motion $x(t)$, $v(t) = x'(t)$ does not imply that the right-hand side of (8) is continuous. However, in this research we are concerned with the existence and mechanical properties of motions affected by higher order derivatives on the right-hand side. With this issue in mind and in order to get clear of other issues and complications caused by possible discontinuities [17], we assume henceforth that the function $F^*(\ldots)$ in (8) and all its entries including all higher order derivatives are continuous on $[0, T)$, $T \le \infty$. In this case, Eq. (8) is mathematically identical, by Lemma 5.1, to the similar equation with all right derivatives, and we assume, for the same reasons, that this equation with all right derivatives has no singular solutions, is solvable for the highest derivative, and in its normal form

$$x^{(k)}(t) = \varphi(t, x, x', \ldots, x^{(k-1)}), \quad t \in [0, T),\ k \ge 2 \tag{19}$$

the function $\varphi(.)$ of (19) satisfies the standard conditions that guarantee the existence, uniqueness and extendibility of solutions over the entire interval $[0, T)$. Under these regularity conditions, there is a unique solution of (19) which depends on the initial data

$$x(0) = x_0, \qquad x'(0) = v_0, \qquad x''(0) = p_2, \ldots, x^{(k-1)}(0) = p_{k-1}, \tag{20}$$

where x_0, v_0 are given and the values $p_2, \ldots, p_{k-1}$ can be considered as control parameters. Since derivatives in $F^*(.)$ of (8) are, in fact, left derivatives, one has to assign initial values for p_2 and $p_k = x^{(k)}(0)$ in such a way that (8), (19) hold for $t = 0$:

$$p_2 = F^*(0, x_0, v_0, p_2, \ldots, p_{k-1}, p_k), \qquad p_k = x^{(k)}(0), \quad k \ge 2, \tag{21}$$

which we call the *consistency condition*. If $k = 2$ and $x''^{-}(t)$ actually enters $F^*(.)$, then there are no free control parameters, due to (21), and the same if $F^*(.)$ does not contain higher order derivatives which renders the usual 2nd order equation with two initial conditions in (8). If $k > 2$, then there are exactly $k - 2$ free control parameters in (20) plus two initial conditions x_0, v_0 for the total of k initial conditions as required by the theory of ODEs. For example, if $k = 3$, then from (21), we compute $p_2 = h(x_0, v_0, p_3)$, and in (20), we obtain $p_{k-1} = x''(0) = p_2 = h(x_0, v_0, p_3)$, as required, whereby p_2 is the initial condition for (19) depending on a free parameter p_3 which also defines initial data $x''^{-}(0) = p_2 = h(x_0, v_0, p_3)$ and $x'''^{-}(0) = p_3$ in (8). If $F^*(\ldots)$ of (8) is linear in higher order derivatives, $k \ge 2$, the calculations are simple, see Section 7.1 below and examples in [13,14]. It is clear that the consistency condition is, in fact, the condition of causality.

7. Fields of effective forces and the parallelogram law

Eq. (19) with initial data (20) and consistency condition (21) has a unique solution in the form

$$x(t) = \xi(t, t_0, x_0, v_0, p_2, \ldots, p_{k-1}), \quad t \in [t_0, T),\ t_0 \geq 0, T \leq \infty, \qquad (22)$$
$$x(t_0) = \xi(t_0, .) = x_0, \qquad dx(t_0)/dt = d\xi(t_0, .)/dt = v_0.$$

The second derivative of this solution defines the function

$$f(t, t_0, x_0, v_0, p_2, \ldots, p_{k-1}) = d^2\xi/dt^2 = x''(t), \quad t \in [t_0, T). \qquad (23)$$

With this function, we can write the equation of motion (8) in the usual form of Newton's second law as $x'' = f(t, \ldots)$. For this reason, we call $f(t, \ldots)$ the *effective force*.

Consider (8) as a vector equation. At the initial moment $t = t_0$, the vector $F^*(t_0, .)$ of (8) defines the vector $F_0 = F^*(t_0, x_0, v_0, p_2, \ldots, p_k)$ due to (20)–(21). If the solution (22) is known, then the vector

$$F^*(t, .) = F^*(t, \xi, \xi', \ldots, \xi(k)) = x''(t) = f(t, t_0, x_0, v_0, p_2, \ldots, p_{k-1}), \quad t \in [t_0, T) \qquad (24)$$

is also specified and equal to the effective force $f(t, \ldots)$ for each $t \in [t_0, T)$.

Imagine that Eq. (8) is integrated for all possible initial data in (20)–(21). Then we have all possible solutions (22) which create a *field of effective forces* $f(t, \ldots)$, see (23), (24), identical to the field $F^*(t, x, x'^-, x''^-, \ldots, x^{(k)-})$ in (8) with respect to its action on a moving body $m(t)$ in (6)–(8). The field $f(t, \ldots)$ does not depend on higher order derivatives implying that over this field of effective forces, Newton's second law has the same form as described by Newton [1] and symbolically specified in (5), (6). This means that effective force (23), (24) embodies "the motive force impressed" mentioned by Newton in his Law II. The original feedback relation (8) represents a force in the sense of Newton only on curves of (22), that is, for such higher order derivatives of $x(t)$ that correspond to parametric equation (22). Outside those curves, i.e., with unrelated $x, x'^-, x''^-, \ldots, x^{(k)-}$ considered as free or partially free parameters, Eq. (8) does not represent any mechanical motion at all.

This observation means that the inclusion of left higher order derivatives on the right-hand side of (8), i.e., application of controls with higher order derivatives (which are *measured or computed* derivatives, thus, automatically *left* derivatives), does not violate any of Newton's laws, if we consider the trajectories defined by (20)–(23). With higher order derivatives, relation (8), due to Lemma 5.1 and assumed solvability of (8) with respect to its higher order derivative, introduces a field of effective forces $f(t, \ldots)$ over which a body moves along the curves (22) as if acted upon by the genuine Newton forces. Therefore, the application of the parallelogram law (Corollary I in [1], also called Law IV of Newton) to the right-hand side of (8) with respect to the vector $F^*(.)$ is incorrect, as indicated in [4]; this is understandable since the right-hand side $F^*(.)$ is, in general for $k > 1$, *not* a force in the sense of Newton, but a feedback liaison of higher order defining certain motions in space for which the vector $F^*(t, x, x'^-, \ldots, x^{(k)-})$ of (8) does *not* define an acceleration, but the vector $f(t, \ldots) = d^2\xi/dt^2 = x''(t)$ defines it.

Fields of effective forces exist also if Eq. (8) contains terms with natural time delays due to finite speed of information transmittal. Effective forces are recovered after the integration of Eq. (8) and act along its solutions obtained with consideration of time delays if they are known. If delays are bounded but not exactly known, then corresponding bands can be evaluated within which the real trajectories are located with effective forces acting along those trajectories. A method of integration in this general case is demonstrated in Section 11, Case 3.

7.1. *Verification of the parallelogram law for effective forces*

Consider the motion of a mass $m = const > 0$ in a plane $x_1 0 x_2$ defined by differential equations in the form (8) with initial conditions $x_i(0) = 0$, $x_i'(0) = 0$, $i = 1, 2$:

$$mx_1'' = a_1 + u_1(t) = a_1 - b_1 x_1''^- - c_1 x_2''^- = F_1, \quad t \geq 0, \qquad (25)$$

$$mx_2'' = a_2 + u_2(t) = a_2 - b_2 x_1''^- - c_2 x_2''^- = F_2, \quad t \geq 0, \qquad (26)$$

where a_i, b_i, c_i are constants. Equating left and right derivatives, we can write the system in the form

$$(m + b_1)x_1'' + c_1 x_2'' = a_1, \quad t > 0, \qquad (27)$$

$$b_2 x_1'' + (m + c_2)x_2'' = a_2, \quad t > 0. \qquad (28)$$

Setting $t = 0$ in (25)–(26) defines the values $u_i(0)$ and the consistency conditions $p_{2i} = x_i''^-(0),\ i = 1, 2$, of (21) which can be determined from (27)–(28) assuming that its principal determinant is nonzero. Determinants are:

$$D = (m + b_1)(m + c_2) - b_2 c_1 \neq 0, \qquad D_1 = a_1(m + c_2) - a_2 c_1, \qquad D_2 = (m + b_1)a_2 - b_2 a_1,$$

so we have $x_1'' = D_1/D$, $x_2'' = D_2/D$, and with zero initial data, the solutions are:

$$x_1(t) = t^2 D_1/2D, \qquad x_2(t) = t^2 D_2/2D, \quad t \geq 0, \qquad (29)$$

yielding a straight line trajectory in the plane $x_1 0 x_2$ with the angle

$$\tan\theta = x_2''(t)/x_1''(t) = D_2/D_1 = const, \quad \text{if } D_1 \neq 0, \quad \text{or}$$
$$\tan\theta = x_1''(t)/x_2''(t) = D_1/D_2 = const, \quad \text{if } D_2 \neq 0. \tag{30}$$

According to Newton's second law, this line should be the line of "the motive force impressed''. If we consider the right-hand sides F_1, F_2 of (25)–(26) as components of the motive force $F(t) = (F_1, F_2)$ before integration, then $F(t)$ would be undefined for $t > 0$, since accelerations on the left-hand sides of (25)–(26) are yet unknown. If we consider right-hand sides of the transformed system (27)–(28) as components of the force, then its direction would be $\tan\beta = a_2/a_1 \neq \tan\theta$, or $\tan\beta = a_1/a_2 \neq \tan\theta$, so it is not "the motive force impressed'' in the sense of Newton's second law. However, if we consider F_1, F_2 in (25)–(26) as components of the effective force $f(t, .)$, *after* the integration of Eqs. (25)–(26), then we have, due to (29) substituted in (25)–(26):

$$F_1 = a_1 - b_1 x_1''^{-} - c_1 x_2''^{-} = a_1 - b_1 D_1/D - c_1 D_2/D = mD_1/D = const,$$
$$F_2 = a_2 - b_2 x_1''^{-} - c_2 x_2''^{-} = a_2 - b_2 D_1/D - c_2 D_2/D = mD_2/D = const,$$

yielding "the direction of the right line in which that force is impressed'' (Law II):

$$\tan\beta = F_2/F_1 = (a_2 - b_2 D_1/D - c_2 D_2/D)/(a_1 - b_1 D_1/D - c_1 D_2/D) = D_2/D_1 = \tan\theta,$$

identical to the line in (30) of the "change of motion'' (Law II) according to (29), in full compliance with Newton's second law of motion. This demonstrates that effective forces obey the parallelogram law. Clearly, the same is valid under any initial conditions since they are eliminated by derivation of variables in (29). It also shows that causality embodied in the consistency condition (21) is essential, since otherwise $u(0)$ would be undefined and the motion in (25)–(26) could not start.

This result, originally proven for linear systems in [14, pp. 35–39], is valid also for a nonlinear system considered over infinitesimally small intervals of time on its discretized trajectory. It demonstrates the validity of the 2nd law of Newton and the parallelogram law for the *effective* forces defined by differential equations of motion containing the *left* and *delayed higher order* derivatives on the right-hand sides.

8. Application to computerized autopilot design

If we ride in a car, or train, or plane, moving at a constant vector velocity $v(t) = const$, we do not feel the speed $|v| = x'(t) = const$, and all measurement devices do not feel it too. This is the universally recognized principle of relativity used with great success by Einstein in [9], see also [10, pp. 1556–1558]. What we really feel are the changes of velocity: the acceleration $x''^{-}(t)$, its variations $x'''^{-}(t)$, and, maybe, its higher order *left* time-derivatives. In reality, the speed $x'(t)$ is undetectable by any devices inside a vehicle, but its left derivatives are detectable and can be used for the control of motion, if included in the properly designed autopilot.

Now we reproduce excerpts from three articles published in Montreal Gazette in 2011.

1. March 18, page B4. "**Airbus faces probe over 2009 crash** (Reuters) **Paris**-

   European plane maker Airbus was placed under investigation on Thursday for the 2009 crash of a flight between Rio de Janeiro and Paris that killed 228 people....

   Investigators are trying to establish why the Airbus 330 plane, operated by Air France, plunged into the Atlantic during a storm on the night of May 31, 2009, killing passengers from 32 nations, including 72 French citizens.

   An investigating judge informed ...the company's lawyers that Airbus was being officially placed under investigation at a meeting in Paris' palace of justice....

   A series of search operations in the Atlantic, the fourth since the crash, are scheduled for March 20 along the Brazilian coast. The $12-million cost will be covered by Airbus and Air France..." (All names and unnecessary details are omitted).

2. May 24, page A17. "**Failed sensors blamed for Air France crash** (Bloomberg News)

   Air France flight 447's black boxes show the plane lost speed and stalled after its air speed sensors failed with the two co-pilots at the controls, two people with knowledge of the investigation said. Chief pilot ...was away from the cockpit when the Airbus A330's airspeed sensors failed, causing the autopilot to disengage over the Atlantic, the sources said. France's BEA air-accident investigation bureau said it plans a preliminary statement on May 27 on the "factual circumstances'' of the June 1, 2009, crash. The failure of the sensors, or *Pitot* tubes, occurred while the plane was cruising at about 35,000 ft, four hours after takeoff from Rio de Janeiro. All 228 passengers and crew aboard the Paris-bound flight perished''.

3. On May 28th, another article appeared in Montreal Gazette, p. A-24 (Reuters) entitled

   "**Aircrew actions scrutinized**'', with the following information: "...The BEA said the reading of the black boxes suggested the crew were not able to determine how fast the plane was flying. That echoes earlier findings which suggest the speed sensors on the plane may have become iced up...''.

This catastrophe could have been avoided if, in addition to outboard *Pitot* tubes that measured the relative velocity v^* of the plane with respect to the wind (which tubes failed in the storm), the *onboard* accelerometers were used to feed the computerized autopilot simultaneously with the tubes (to check their proper operation) and separately, in case of the tubes failure, in order to compute the velocity values by the formula

$$v(t) = v_0 + \int_0^t a(s)ds, \qquad a(s) = v'^{-}(s) = \lim\ [v(s) - v(s - \Delta s)]/\Delta s, \qquad s - \Delta s \in (0, t), \Delta s \to +0, \tag{31}$$

where $v_0 = v^*$ is the average value of the relative velocity supplied by the tubes right before their failure and $a(s) = v'^{-}(s),\ s \in (0, t)$, is the horizontal acceleration supplied to the autopilot by the *onboard* accelerometer *independently* of the outboard tubes. This is *not* the right derivative calculated by the standard formula (2) with $x'(t)$ instead of $v(t)$ but the real physical acceleration data supplied by the measuring device directly to the autopilot which data already contain a small time uncertainty always present in real physical measurements. Later, when the tubes may unfreeze in higher temperatures or at a lower altitude, they could be put in operation again, in addition to the accelerometers.

Unfortunately, such design is absent in the textbooks on mechanics and engineering since the acceleration and the higher order derivatives of velocity are not considered as possible arguments of the forces in the current mathematical representation of the 2nd law of Newton. This omission can and should be corrected. Nature takes heavy toll for mistaken theories presented in some textbooks. The block (31) must be incorporated into the autopilots and control panels of airplanes and spacecrafts to assure the safety and the proper control of the vehicle.

9. Stability in the reactive motion of a spacecraft

The ascending vertical motion of a rocket with the axis 0*x* directed straight up was considered by Mestschersky in 1897 and described by the equation [7, p. 114, Eq. (1)]:

$$mx''(t) = -mg + \sigma(p^* - p_x) - m'(t)w^* - R(x'(t)). \tag{32}$$

Here, $m(t)$ is the variable mass of the rocket, $x(t)$ is its vertical coordinate (height) and $g = 9.8\ \mathrm{m/s^2}$ is the gravitational acceleration; σ is the area of the funnel opening that ejects burnt gases, p^* is the pressure of ejected gases, p_x is the air pressure at the height $x(t)$; $m'(t) < 0$ is the rate of change of mass of the rocket due to combustion, and $w^* = v - w$ is "*geometric difference between velocities of separating mass and the body directed straight down*" [7, pp. 113–114] where w is the absolute velocity of "*separating mass*" (gases) and $v(t) = x'(t)$ is the absolute velocity of the rocket; $R(x'(t))$ is the resistance of the air.

A solution of Eq. (32) is given in [7, pp. 114–115], assuming the resistance of the air $R(x') = R(v) = a + bv$, uniform combustion $m = m_0(1 - \alpha t) \geq m^* > 0, \alpha > 0$ over some time $0 \leq t \leq T = (1 - m^*/m_0)/\alpha$, where m_0, m^* are initial and final masses of a rocket, and the constancy of parameters: $w^* = const$ and $p = \sigma(p^* - p_x) = const$. At higher velocities of a rocket, the air resistance is quadratic, $R(x') = R(v) = av^2$. With a finite volume of fuel in a rocket, Eq. (32) and also (33)–(36) below are valid over finite periods of time when combustion takes place, and over periods of free flight, one has to set $m'(t) \equiv 0$, $p = 0$ in (33) returning to Newton's Eq. (1) or its generalization (8) with mass $m = const$, different and differently distributed over different periods of free flight. Many other examples of motion with variable masses are presented in [7].

To consider the acceleration assisted control, see Section 4 above, we adopt, for simplicity, the assumptions of Mestschersky, except for the resistance of air which we take in the form $R(x') = R(v) = av^2$. With the notation $x'(t) = v(t)$, this renders a differential equation of the first order (Riccati equation) for $v(t)$:

$$m(t)v'(t) = -m(t)g + p - m'(t)w^* - av^2, \quad 0 \leq t \leq T. \tag{33}$$

Clearly, the same equation governs the launch of a space shuttle, but with a difference. A rocket is launched like a bullet, but a shuttle ascends slowly which can be seen on T.V. showing a shuttle launch. This is due not only to a greater weight of a shuttle, but mainly to the presence of humans in it. Indeed, the health of humans requires certain gravitational conditions with total acceleration $v'(t) + g$ in the range $[rg, ng]$, where the numbers $0 < r \leq 1$, $1 \leq n \leq n^* < 9$ depend on personal health and flight duration, and are determined by medical considerations since $v'(t) + g > n^*g$ or $|v'(t) + g| < rg$ for a long time may cause sickness and incapacity of a person in the shuttle.

This means that $m'(t)$, if used to control dangerous gravitational load on people in the shuttle, must depend on *acceleration* $v'(t)$. Suppose that $m'(t) = -b + qv'$, where b, q are positive constants. With small q, we have $m'(t) < 0$, thus $dm < 0$, due to combustion, so that $qv' > 0$ acts as regulator and moderates the thrust in order to prevent too high accelerations. Substituting the feedback $m'(t) = -b + qv'$ into (33) and assembling the terms that do not depend on $v(t)$, we obtain the equation with control parameter q:

$$m(t)v'(t) = h(t) - qw^*v'(t) - av^2(t), \qquad h(t) = -m(t)g + p + bw^*. \tag{34}$$

However, the feedback $qw^*v'(t)$ in (34) depends on *measured* acceleration which carries a small time delay $\delta > 0$, yielding the equation

$$m(t)v'(t) = h(t) - qw^*v'(t - \delta) - av^2(t). \tag{35}$$

This is *not* an ordinary delay differential equation, DDE, since delay affects the highest order derivative, and there is no theory yet for such equations. An attempt to formally expand $v'(t-\delta)$ into Taylor series taking the first two terms in it to obtain a normal ODE, yields an equation with a small parameter at the highest derivative, and the method fails. Indeed, we have $v'(t-\delta) = v'(t) - v''(t)\delta + v'''(t)\delta^2 - \cdots$, rapidly converging for small δ if all derivatives are uniformly bounded. Putting the first two terms into (35) for $v'(t-\delta)$, we get the equation

$$qw^* v''(t)\delta = m(t)v'(t) - h(t) + qw^* v'(t) + av^2(t), \tag{36}$$

which may be extremely unstable. Indeed, assuming that the length of information transmittal is 1 mm and its speed equals the speed of light, we have $\delta = 0.1\ \text{cm}/3 \times 10^{10}\ \text{cm/s} \cong 0.3 \times 10^{-11}$ s, so that the rate of change in acceleration $v''(t) = dv'/dt \cong 10^{11}[\ldots]$, the bracket standing for the right-hand side of (36) divided by qw^*, which, if nonzero, would cause the acceleration to explode. To illustrate this effect, consider an example obtained from (35) by setting $m(t) \cong 1$, $h(t) \cong 2$, $qw^* = 1$, $a = 0$, yielding an equation similar to (35) but much simpler:

$$v'(t) = 2 - v'(t-\delta),\ v(0) = v_0. \tag{37}$$

If $\delta = 0$, then $v'(t) = 1$ and the solution is $v(t) = v_0 + t$. If $\delta \neq 0$ small, then, using the first two terms of the Taylor series above, we obtain the equation $v' = 2 - v' + \delta v''$, that is, $\delta v'' - 2v' + 2 = 0$. For this equation of the second order, we have to add one more initial condition, and to comply with (37) for $\delta = 0$, $t = 0$, we should set $v'(0) = 1$. The characteristic equation is $\delta r^2 - 2r + 2 = 0$, with roots $r_{1,2} = [1 \pm (1-2\delta)^{0.5}]/\delta$. For small $\delta \cong 10^{-11}$, we have $(1-2\delta)^{0.5} = 1 - \delta + \delta_2 - \cdots$, yielding $r_1 = (2-\delta)/\delta \cong 2/\delta,\ r^2 = (\delta - \delta^2)/\delta = 1 - \delta \cong 1$, and the general solution is $v(t) = a\exp(2t/\delta) + be^t$. Using initial conditions $v(0) = a + b = v_0,\ v'(0) = 2a/\delta + b = 1$, we get $a = \delta(1-v_0)/(2-\delta) \cong \delta(1-v_0)/2$, $b = (2v_0 - \delta)/(2-\delta) \cong v_0$ so that $v(t) \cong 0.5\,\delta(1-v_0)\exp(2t/\delta) + v_0 e^t \to \infty$, and very fast for $\delta \cong 10^{-11}$ if $v_0 \neq 1$.

Hence, the Taylor series approach to control the ascent of spacecrafts considering small nonzero delays in the measured information is impractical and theoretically unacceptable. The acceptable solution is to ignore such delays and set $\delta = 0$, which renders one and the same equation in (34)–(36) with acceleration assisted control whose action provides a smoothing effect on the flight with a seemingly increased mass of the spacecraft corresponding to actually decreased acceleration and lesser gravitational load on the people in the spacecraft. However, in some other cases, the time delays in the transmission of information must be accounted, and a method to do it is presented in Section 11, Case 3.

10. Generalization of the Lagrange and Hamilton equations

To complete the inclusion of the left higher order derivatives into the forces and the generalized equations of theoretical and applied mechanics, let us consider differential systems of Newtonian mechanics in relation to variable masses and left higher order derivatives on the right-hand sides. First, we reproduce some well known concepts of analytical mechanics [2–4] with *constant* masses and *without* higher order derivatives, thus, considering only the *geometry* of Newtonian motion as presented in classical theory.

Newtonian equations of motion for a constrained mechanical system of N point-wise masses are written in the form:

$$m_i x_i'' = F_i(t, x, v) + R_i(t, x, v), \quad i = 1, \ldots, N; \qquad x'' = v' = d^2x/dt^2, \quad v = dx/dt,\ t \geq 0,\ x \in R^{3N}, \tag{38}$$

where m_i are constant masses, F_i are active forces and R_i are reactions of constraints acting on masses m_i. The variables x_i, v_i, x_i'' are state, velocity and acceleration vectors in the Cartesian (rectangular) coordinate system (phase space). Since the mass m_i can be subject to forces acting from other masses, the $3N$-vector x composed of N 3D-vectors x_i (that denote coordinates of masses m_i) is included in the forces F_i and R_i together with velocity v. This is a short form to avoid a double index writing $m_i x_i'' = F_i(t, x_k, v_k) + R_i(t, x_k, v_k)$, cf. (42) below, where x_k means $\{x_1, \ldots, x_N\}$ with index k not included in subsequent summations. If Eqs. (38) are divided by masses and reduced to the normal form by writing dv_i/dt instead of x_i'' with vector equations $dx_i/dt = v_i$ added, we obtain $6N$ dimensional vector equation (38) of the first order, whereby F_i can be regarded as controls (or containing controls). Constraints are assumed *ideal* which means that the total work of constraint reactions is zero, $\Sigma R_i \delta x_i = 0$, where δx_i are any possible, i.e., allowed by the constraints (virtual, t fixed) displacements. Using this equation to exclude the unknown reactions of constraints yields the general equation of motion (principle of D'Alembert):

$$\sum_{i=1}^{N} [m_i x_i'' - F_i(t, x, v)]\delta x_i = 0, \tag{39}$$

for the N-mass system (38). At rest $x_i'' \equiv 0$, and in this case, Eq. (39) renders the criterion (necessary and sufficient condition) for the equilibrium of active forces F_i (principle of virtual displacements, J. Bernoulli, 1717).

At the time of Newton (1687) [1], and for more than two centuries thereafter, only constant masses m_i were considered, with "the motive force impressed" $F_i(.)$ being known functions of time, coordinates and velocities. It means that trajectories of motion were considered as fixed geometric curves $x(t)$ known for all $t \in [0, T)$. Time t was perceived as absolute, and reactive forces were irrelevant with the consideration of constant masses. Since the results of Buquoy (1815) [6] were not properly recognized, thus, unknown for more than a century, and reactive forces were ignored even after the publications of Mestschersky [7] and Levi-Civita [8], it is not surprising that Eqs. (38), (39) are still written in their maiden form as in the eighteenth century, without reactive forces nor control forces depending on left higher order derivatives.

With the advent of jet propulsion, the forces in (38)–(39) should be replaced by $F^*(.)$ of (7), with variable masses $m_i(t)$, which would, of course, alter the classical formulas (38)–(39). With the use of acceleration assisted control, there is no problem, if left and right time derivatives are considered identical and equations of motion are resolved for actual accelerations as in (38), requiring nonzero Jacobian with respect to accelerations. For controls with the *left higher order* derivatives, Eqs. (38)–(39) are propelled by the *effective* forces generated by the expressions $F_i(.)$ in (38) with ideal reactions $R_i(.)$ excluded by (39). It means that $F_i(.)$ can still be formally written in (39) with higher order derivatives of $v(t)$ included therein for subsequent integration; this would render the trajectories of motion (22) and identify the actual effective forces (23)–(24) in the system which forces are to be recognized as "the motive force impressed" mentioned by Newton in his Law II [1].

The general equation of motion (39) excludes *reaction forces of ideal* constraints, but *not* the constraints themselves which are still restricting coordinates and velocities of the motion, no matter if forces of reaction are excluded. Let us consider the exclusion of constraints altogether, i.e., elimination of kineto-statical relations of the problem [2].

Analytically, constraints are expressed by several independent equations:

$$l_k(t, x_i, v_i) = 0, \quad k = 1, \dots, s. \tag{40}$$

If Eqs. (40) do not contain velocities v_i or can be integrated not to contain them, the constraints are called *geometric*, and the system (38)–(40) is called *holonomic*. In this case, from equations $l_k(t, x_i) = 0$ one can express certain s coordinates as functions of $3N - s$, other coordinates and time t and consider those $3N - s$ coordinates as independent variables that define the state of the system at time t. However, it is not binding to take Cartesian coordinates as independent variables. It may be convenient to express all $3N$ Cartesian coordinates as functions of $n = 3N - s$ independent parameters $q_1, \dots, q_n$ and time t which define the so called *configuration* space. Substituting thus obtained $x_i = x_i(t, q_1, \dots, q_n)$ into (38) and using (39), a new system of second order equations with respect to independent parameters $q_1, \dots, q_n$ and time t is derived. Those parameters (called *generalized* coordinates) do not have the transparent meaning of Cartesian coordinates, but they present a differential system without constraints, of *minimal* order with respect to independent variables $q_i(t)$ which define all Cartesian (rectangular) coordinates and velocities, thus, the state of original system (32) subject to constraints (40). Using (39), and noting that the elementary work of active forces $\delta A = \sum_{j=1}^{N} F_j \delta x_j = \sum_{i=1}^{n} Q_i \delta q_i$, the minimal order system of the *Lagrange equations of the second kind* is obtained in the form:

$$\frac{d}{dt}\frac{\partial T}{\partial q_i'} - \frac{\partial T}{\partial q_i} = Q_i, \qquad Q_i = Q_i(t, q_k, q_k'), \qquad T = 1/2 \sum a_{ik} q_i' q_k' + a_i q_i' + a_0, \quad i = 1, \dots, n. \tag{41}$$

Here on the left stand generalized forces of inertia expressed through kinetic energy T of the system, Q_i are generalized active forces, and sums are taken from 1 to n. It is clear that Lagrange's equations on the left in (41) represent Newton's second law of motion (38) in generalized coordinates $\{q_k\}$, with constant masses and with reactions R_i excluded by (39). If constraints are stationary, i.e., (40) does not depend on t, then a_0, a_i are zero in (41). Substituting the expression of T in (41) into Lagrange equations yields

$$\sum_{k=1}^{n} a_{ik} q_k'' + (.) = Q_i(t, q_j, q_j'), \qquad q_i'' = Q_i^*(t, q_j, q_j'), \quad i = 1, \dots, n, \tag{42}$$

where (.) stands for terms not containing second derivatives, and the second equation is the unique solution of the first one for q_i'', since the determinant $det\ (a_{ik})_{i,k=1}^{n} \neq 0$. This is known as the explicit form of Lagrange's equations [2, pp. 39–40] which define the motion of the system determined by initial values $q_i(0)$, $q_i'(0)$.

The transformation to independent generalized coordinates in configuration space that excludes geometric constraints does not depend on the presence of higher order derivatives in (39). For holonomic systems with variable masses, the generalized coordinates $q_i(t)$ can be introduced to eliminate the constraints (40) even if forces in (39) depend on higher order derivatives. It is interesting and important that the equations thus obtained will also be in the form of Lagrange's equations but with quite different entries, and the order of those equations will be minimal, by construction, with generalized forces depending on left higher order derivatives of generalized coordinates and with kinetic energy corresponding to variable masses.

Indeed, according to (5), (8) and (39), we have after the transformation:

$$\sum_{i=1}^{N} \{m_i[v_i(t, q_1, \dots, q_n)]' + (v_i - w_i)m_i' - F_i^*(t, x, x'^-, x''^-, \dots, x^{(k)-})\}\delta x_i(t, q_1, \dots, q_n) = 0, \tag{43}$$

where $v_i(.) = x_i'(.), x^{(r)}- = [x(t, q_1, \dots, q_n)]^{(r)-}$, $r = 0, \dots, k$, but $\delta x_i(t, q_1, \dots, q_n)$ are not arbitrary, due to (40). However, all $x_i(t, q_1, \dots, q_n)$, $x^{(r)-} = [x(t, q_1, \dots, q_n)]^{(r)-}$, and $\delta x_i(t, q_1, \dots, q_n)$ can be expressed through $q_j, q_j', \dots, q_j^{(k)-}$ and δq_j $(j = 1, \dots, n)$ yielding

$$\sum_{i=1}^{N} \{m_i[v_i(t, q_1, \dots, q_n)]' + (v_i - w_i)m_i' - F_i^*(t, q, q'^-, q''^-, \dots, q^{(k)-})\} \sum_{j=1}^{n} (\partial x_i/\partial q_j)\delta q_j = 0,$$

where we retain the same notation for $F_i^*(q,\ldots)$ derived from $F_i^*(x,\ldots)$ of (43) after the transformation. Changing the order of summation, we obtain

$$\sum_{j=1}^{n} \delta q_j \sum_{i=1}^{N} \{m_i v_i' + (v_i - w_i)m_i' - F_i^*(t, q, q'^-, q''^-, \ldots, q^{(k)-})\}(\partial x_i/\partial q_j) = 0. \tag{44}$$

Since δq_j in (44) are arbitrary virtual displacements, we have

$$\sum_{i=1}^{N} \{m_i v_i' + (v_i - w_i)m_i' - F_i^*(t, q, q'^-, q''^-, \ldots, q^{(k)-})\}(\partial x_i/\partial q_j) = 0, \quad j = 1, \ldots, n. \tag{45}$$

Let us call "generalized forces" the expressions:

$$\sum_{i=1}^{N} [F_i^*(t, q, q'^-, q''^-, \ldots, q^{(k)-}) + w_i m_i'(t)](\partial x_i/\partial q_j) = Q_j^*, \quad j = 1, \ldots, n, \tag{46}$$

so that (45) can be rewritten in the form

$$\sum_{i=1}^{N} [m_i(t) v_i(t, q_1, \ldots, q_n)]'(\partial x_i/\partial q_j) = Q_j^*, \quad j = 1, \ldots, n. \tag{47}$$

The term $[m_i v_i]'$ $(\partial x_i/\partial q_j)$ under summation sign in (47) can be transformed as follows:

$$[m_i v_i]'(\partial x_i/\partial q_j) = d[m_i v_i\,(\partial x_i/\partial q_j)]/dt - m_i v_i d\,(\partial x_i/\partial q_j)/dt. \tag{48}$$

Since $v_i = dx_i/dt = \partial x_i/\partial t + \Sigma\ (\partial x_i/\partial q_j)q_j'$, so $\partial v_i/\partial q_j' = \partial x_i/\partial q_j$. Also, we have

$$\partial v_i/\partial q_k = \partial^2 x_i/\partial t\partial q_k + \Sigma\ (\partial^2 x_i/\partial q_j \partial q_k)q_j' = d(\partial x_i/\partial q_k)/dt. \tag{49}$$

Relations (48)–(49) imply that

$$\begin{aligned}[m_i v_i]'\ (\partial x_i/\partial q_j) &= d[m_i v_i\ (\partial x_i/\partial q_j)]/dt - m_i v_i\ (\partial v_i/\partial q_j)\\ &= d[\partial(0.5 m_i v_i^2)/\partial q_j']/dt - \partial(0.5 m_i v_i^2)/\partial q_j.\end{aligned} \tag{50}$$

Summing up the expression in (50) to obtain the left-hand side of (47), we get

$$\begin{aligned}\sum_{i=1}^{N}[m_i v_i]'(\partial x_i/\partial q_j) &= d\left[\partial\left(0.5\sum_{i=1}^{N} m_i v_i^2\right)\Big/\partial q_j'\right]\Big/dt - \partial\left(0.5\sum_{i=1}^{N} m_i v_i^2\right)/\partial q_j\\ &= d(\partial T^*/\partial q_j')/dt - \partial T^*/\partial q_j = Q_j^*, \quad j = 1, \ldots, n\end{aligned} \tag{51}$$

$$T^* = 0.5\Sigma m_i(t) v_i^2(t, q_1, \ldots, q_n).$$

Relations (51) *have the form of Lagrange's equations* (41), *with a difference*:

(1) "*generalized forces*" Q_j^* *may depend on higher order derivatives of generalized coordinates*, $q_j^{(k)-}$, *thus, the explicit form* (42) *of Lagrange's equations is not preserved*;
(2) *the function* $T^* = 0.5\sum m_i(t)v_i^2(.)$ *in* (51) *corresponds to variable masses with the same formula as in the case of constant masses, but* $T^* \neq T(.)$ *of* (41);
(3) *the function* T^* *which resembles the expression of kinetic energy may not represent the real kinetic energy of the system. To determine the real kinetic energy of the system, one has to equate left and right derivatives in* (46), (51), *solve the higher order system with additional initial conditions to obtain the solution in the form of* (22), *compute the effective velocities* v_i^*, *not those* v_i *that appear in* (43)–(51), *and compute the real (effective) kinetic energy of the system.*

Preservation of the form of Lagrange's equations for holonomic systems with variable masses and forces depending on the left higher order derivatives on the right-hand sides presents a useful method allowing one to formally construct the equations of motion excluding geometric constraints, solve the resulting differential system of minimal order with respect to independent generalized coordinates in the configuration space, then return to natural rectangular coordinates, velocities and accelerations, and evaluate by (23) the effective forces in the system that represent Newtonian forces [1] in this case. The reader can check that, with $m = const$ and higher order derivatives absent from (43), Eqs. (51) coincide with Lagrange's equations (41)–(42).

If in Lagrange's equations (41) generalized forces Q_i do not depend on generalized velocities, $Q_i = Q_i(t, q_1, \ldots, q_n)$, then there exists a potential function $P(t, q_1, \ldots, q_n)$ such that $Q_i = -\partial P/\partial q_i$. Introducing *kinetic potential (the Lagrange function)* $L = T - P$, the system (41) on the left, with constant masses, can be written in the form:

$$\frac{d}{dt}\frac{\partial L}{\partial q_i'} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \ldots, n. \tag{52}$$

If active forces in (43) depend on the left higher order derivatives, so do the generalized forces $Q_j^*(.)$ in (46), (47), (51). In this case, consider the "generalized" potential function $V(t, q, q'^-, q''^-, \dots, q^{(k)-})$, if it exists, such that $Q_i^*(.)$ of (46) can be expressed by the formulas, cf. [2, p. 44]:

$$Q_i^* = d(\partial V/\partial q_i')/dt - \partial V/\partial q_i, \quad i = 1, \dots, n. \tag{53}$$

With Q_i^* from (53), Eqs. (51) can be written in the form (52) if L is substituted by the function $L^* = T^* - V$ with T^* from (51). These new equations are not the second order equations, but a higher order system written in the form (52). This form is preserved in mechanical systems with variable masses and higher order derivatives in active forces, if there exists a potential function $V(.)$ with which Q_i^* can be expressed in the form (53). An example of such potential is furnished by $V = (1 + r'^2/c^2)/r$ which presents the generalized potential in the sense of (53) for Weber's electro-dynamic force of attraction F^* cited in Section 5, see [2, p. 45].

Eqs. (52) suggest new coordinates proposed by Hamilton. Denote $\partial L/\partial q_i' = p_i$ (generalized impulses), so that by (52) $dp_i/dt = p_i' = \partial L/\partial q_i$, and consider new variables $p_1, \dots, p_n$ which together with old variables $q_1, \dots, q_n$ constitute the set of $2n$ variables of Hamilton. Since $\partial^2 L/\partial q_i'\partial q_k' = det\ (a_{ik})_{i,k=1}^n \neq 0$, see expression of T in (41), so the Jacobian of $\partial L/\partial q_i'$ is nonzero, and equations $\partial L/\partial q_i' = p_i$ can be resolved for q_i' yielding $q_i' = \varphi_i(t, q_k, p_k)$, which together with $p_i' = \partial L/\partial q_i = \theta_i\ (t, q_k, p_k)$ present the Hamiltonian system of $2n$ equations of the first order equivalent to the Lagrangian system of n Eqs. (52) of the second order, cf. (38), (42). If the quantity $p_i q_i' - L$ is expressed as a function of $\{t, q_i, p_i\}$ and denoted by H, then equations of motion (52) in the Lagrangian form with constant masses can be represented also in the *Hamiltonian* or *canonical* form as follows [2, pp. 263–264]:

$$\delta H = \delta\{\Sigma p_i q_i' - L\} = \Sigma(q_i'\,\delta p_i - p_i'\,\delta q_i), \quad \text{thus,} \quad q_i' = \partial H/\partial p_i, \quad p_i' = -\partial H/\partial q_i. \tag{54}$$

If instead of L in (52), (54), the function $L^* = T^* - V(t, q, q'^-, q''^-, \dots, q^{(k)-})$, with T^* from (51) is considered, denote $\partial L^*/\partial q_i' = p_i^*(.)$, so that by (52) with L^* instead of L, we have $dp_i^*(.)/dt = p_i^{*\prime} = \partial L^*/\partial q_i$. If the Hessian $\partial^2 L^*/\partial q_i'\partial q_k' \neq 0$, then the Jacobian $\partial L^*/\partial q_i'$ is nonzero, and equations $\partial L^*/\partial q_i' = p_i^*(.)$ can be resolved for q_i' yielding $q_i' = \varphi_i\ (t, q_k, p_k^*(.))$, which together with $p_i^{*\prime} = \partial L^*/\partial q_i = \theta_i\ (t, q_k, p_k^*(.))$ present the generalized Hamiltonian-like system of $2n$ equations of higher order equivalent to the generalized system of n equations (52) with L^* substituted for L. If the quantity $\Sigma p_i^*(.)q_i' - L^*(.)$ is expressed as the function of $\{t, q_i, p_i^*(.)\}$ and denoted by $H^*(.)$, then we see that the canonical form (54) is preserved, *with the understanding that coordinates $p_i^*(.)$ contain $w_i m_i'(t)$ and higher order derivatives* on which $p_i^*(.)$ and Q_i^* of (53) depend, so that *new equations*

$$\delta H^* = \delta\{\Sigma p_i^*(.)q_i' - L^*(.)\} = \Sigma(q_i'\delta p_i^* - p_i^{*\prime}\delta q_i), \quad \text{thus,}$$
$$q_i' = \partial H^*/\partial p_i^*, \qquad p_i^{*\prime} = -\partial H^*/\partial q_i, \tag{55}$$

present a *higher order* system from which the *effective* forces (23) actually acting in the system can be recovered after integration and passage to the Cartesian coordinates.

11. Method of integration for systems with delayed left higher order derivatives

Consider a physical pendulum consisting of a rod OC of length l suspended in a hinge at O with a heavy disk of mass M fixed at its center to the end C of the rod. With such pendulums are equipped free standing clocks that can be seen in furniture or antiquity stores. Friction at the hinge is neutralized by a spring or a battery, and a mass of the rod can be ignored. The moments of inertia of the disk are

$$I_C = \int_o^r r^2 dm = \int_o^r 2\pi\rho r^3 dr = 0.5Mr^2, \qquad I_O = I_C + Ml^2 = 0.5M(r^2 + 2l^2).$$

The pendulum oscillates in a plane xOy with axis Ox directed straight down and axis Oy directed to the right. It is required to derive the equations of motion.

Case 1. *Classical solution*. The system has one degree of freedom, and it is convenient to take the angle φ between Ox and the rod as the generalized coordinate $q = \varphi$. The coordinates of the center of mass are: $x_c = l\cos\varphi$, $y_c = l\sin\varphi$. The acting force of gravity $Mg = (X, 0)$ is directed straight down, so that generalized force $Q = X\partial x_c/\partial\varphi = -Mgl\sin\varphi$. The kinetic energy is $T = 0.5I_O\varphi'^2$, so that $\partial T/\partial\varphi' = I_O\varphi'$, $\partial T/\partial\varphi = 0$, yielding the Lagrange equations (41) and (42) for the case as follows:

$$I_O\varphi'' = Q = -Mgl\sin\varphi, \qquad \varphi'' + 2gl\sin\varphi/(r^2 + 2l^2) = 0, \qquad \varphi(0) = \varphi_0, \varphi'(0) = 0. \tag{56}$$

The equivalent length of the mathematical pendulum with the same period is $l^* = r^2/2l + l$. The potential function for Q can be taken in the form $P = -Mgl\cos\varphi$, so that with the Lagrange function (kinetic potential) $L = T - P = 0.5I_O q'^2 + Mgl\cos q$, $q \equiv \varphi$, the Lagrange equation in (56) at left can be represented in the form (52). If we denote $p = \partial L/\partial q'(\equiv I_O q')$, then, due to (52), we have $p' = dp/dt = \partial L/\partial q = -Mgl\sin q$, and can define the Hamiltonian $H(t, q, p) = pq' - L = I_O q'^2 - L = p^2/2I_O - Mgl\cos q$, yielding canonical equations of the motion, cf. (54):

$$q' = \partial H/\partial p = p/I_O, \qquad p' = -\partial H/\partial q = -Mgl\sin q, \quad q \equiv \varphi, \ p \equiv I_O\varphi', \tag{57}$$

which are equivalent to (56) since $\varphi'' \equiv p'/I_O = -Mgl\sin\varphi/I_O = -2gl\sin\varphi/(r^2+2l^2)$. This classical solution which excludes variable reaction in the hinge can be found in most textbooks on theoretical mechanics.

Case 2. Consider the same pendulum submerged in an aquarium with water. Then the pendulum will be affected by the additional force of water resistance $F = -\alpha\varphi' - \beta\varphi''$ ($\alpha, \beta = const > 0$), where $-\alpha\varphi'$ is the Newtonian fluid friction, and $-\beta\varphi''$ is the Kirchhoff–Thomson adjoint fluid acceleration resistance [11,12], see Section 5. Now we have a different generalized force $Q^* = -Mgl\sin\varphi - \alpha\varphi' - \beta\varphi''$ with the same kinetic energy of the pendulum. This yields a different equation for the same generalized coordinate $q = \varphi$:

$$I_O\varphi'' = Q^* = -Mgl\sin\varphi - \alpha\varphi' - \beta\varphi'', \quad \text{or} \quad (I_O+\beta)\varphi'' + \alpha\varphi' + Mgl\sin\varphi = 0. \tag{58}$$

If $\alpha = 0$, then Eq. (58) can be converted into the canonical form with the introduction of the generalized potential function $V = -Mgl\cos\varphi + \beta\varphi''\varphi$, such that $Q^* = -\partial V/\partial\varphi + d(\partial V/\partial\varphi')/dt$. It is left to the reader to obtain generalized Hamiltonian equations through the introduction of the L^* function with this generalized potential V. Setting also $\beta = 0$, one would return to the classical canonical equations (57). It is interesting and important that the acceleration of a moving body can enter Lagrange's and Hamilton's equations also through generalized forces, not only through kinetic energy which is stipulated by the classical representation (38) of Newton's second law of motion. The preservation of the form of Lagrange's and Hamilton's equations for generalized systems with higher order derivatives on the right-hand sides opens a way for the use of those equations in the large area of soft control with higher order derivatives, excluding ideal constraints whose reactions, if needed, can be found afterward.

Case 3. Consider the same pendulum in the air affected by a strong wind from a ventilator in the direction of the negative axis Oy (to the left). With a laser, small computer and connecting wires, the ventilator can be controlled to supply a flow of air upon the disk from the right to the left generating a force $Y = -(a + b\varphi' + h\varphi'' + k\varphi''') < 0$ depending on higher order derivatives of the motion. The generalized force is

$$Q^* = X\partial x_c/\partial\varphi + Y\partial y_c/\partial\varphi = -Mgl\sin\varphi - (a + b\varphi' + h\varphi'' + k\varphi''')l\cos\varphi, \tag{59}$$

where a, b, h, k are some constants. Since $\varphi'(t)$ is *measured* and φ'', φ''' are also measured or computed from the measured $\varphi'(t)$, so all three derivatives in (59) are necessarily *left* and *delayed* due to a finite speed of information transmittal [18, p. 1344]. In this situation, the time delays may play a major role. To fix the ideas, let us consider, for simplicity, that $|\varphi(t)|$ is small, and also $b = 0,\ h > 0,\ k > 0,\ a > |h\varphi'' + k\varphi'''|$. Then in (59), we can set $\cos\varphi = 1$, and consider

$$Q^* = -al - Mgl\sin\varphi(t) - l[h\varphi''^{-}(t-\delta_1) + k\varphi'''^{-}(t-\delta_2)]. \tag{60}$$

Note that $\varphi(t)$ is without delay since it is *not* a measured and transmitted quantity. With this generalized force and the same kinetic energy, we have the equation, cf. (56), (58):

$$I_O\varphi'' = Q^* = -al - Mgl\sin\varphi(t) - l[h\varphi''^{-}(t-\delta_1) + k\varphi'''^{-}(t-\delta_2)], \quad t \ge 0. \tag{61}$$

In (60), (61), we assume that all three moments of time are within the time interval of the actual motion; out of this interval, the entries are equal zero. Physically, it is clear that always $\delta_1 > 0,\ \delta_2 > 0$; the question is whether we can ignore both or one of them. It is also clear that oscillations will be distorted and not symmetric with respect to the axis *Ox*.

Recall [18, p. 1344] that over the length of 100 cm, information transmittal with the speed of light takes the time $\delta \cong 10^{-8}$ s, whereas information transmittal with the speed $v^* \cong 10^{-2}$ cm/s of the ordered motion of electrons over the same length of 100 cm would take $\delta^* \cong 10^4$ s $= 167$ min $= 2.8$ h, which makes quite a difference. For information transmittal over 1 cm, the corresponding delays are 10^{-10} and 100 s. For different delays δ_1, δ_2 within $[10^{-10}, 1]$ s, different dynamics can be obtained for the same system in (61). Equating left and right derivatives, we consider the following cases.

3.1. If $\delta_2 \cong 10^{-8}$ s, small, and $\delta_1 > \delta_2$, then the differential equation (61) is changing its order and right-hand sides over different intervals, and when it is of the third order, initial conditions in (56) are insufficient to define its unique solution. At $t = 0$, derivatives on the right in (61) are not yet in action, so over $[0, \delta_2)$, we have in (61) the equation as in (56) with the term $-al$ and same initial conditions, yielding the values $\varphi(\delta_2) \cong \varphi_0,\ \varphi'(\delta_2) \cong 0, \varphi''(\delta_2) \cong -(al + Mgl\sin\varphi_0)/I_O$. At $t = \delta_2$, this value $\varphi''(\delta_2)$ presents the initial condition for Eq. (61), where the second derivative on the right is not yet in action. This assures the continuity of the motion over $[0, \delta_1)$ but with the dynamics of the third order over $[\delta_2, \delta_1)$ since the third derivative in (61) comes into play and will overtake the motion for small $\delta_2 \cong 10^{-8}$. At the moment $t^* = \delta_1$, the term $h\varphi''(t-\delta_1)$ on the right in (61) comes into play, so we have to replace the value $\varphi''(\delta_2)$ by the new initial condition at $t = \delta_1$ according to the equation $I_O\varphi''(\delta_1) = -al - Mgl\sin\varphi(\delta_1) - l[h\varphi''(0) + k\varphi'''\ (\delta_1 - \delta_2)]$, see (61), which is the consistency condition (21) for the case, yielding $\varphi''(\delta_1) = -(al + Mgl\sin\varphi\ (\delta_1))/I_O - l[-h(al + Mgl\sin\varphi_0)/I_O + k\varphi'''\ (\delta_1-\delta_2)]/I_O$, where $\varphi'''\ (\delta_1 - \delta_2)$ is known from the preceding segment $[\delta_2, \delta_1)$ of the motion with $h\varphi''^{-}(t-\delta_1)$ in (61) *not* yet in action. Now, for $t \ge \delta_1$, the motion is defined by the third order differential equation, and with the approximation $\delta_2 \cong 0$, this equation can be written as ordinary DDE: $l_k\varphi'''(t) = -I_O\varphi''(t) - al - Mgl\sin\varphi(t) - lh\varphi''(t-\delta_1), t \ge \delta_1$ with $\varphi(\delta_1), \varphi'(\delta_1)$ defined as the end-point values of the previous segment of $\varphi(t)$ over $[0, \delta_1]$ and $\varphi''(\delta_1)$ given by the consistency condition.

3.2. If $\delta_1 \cong 10^{-8}$ s, small, but δ_2 is relatively large, then in (61) we have, in fact, the second order differential equation with discontinuity on the right-hand side. Indeed, until after $t^* > \delta_2$ the third derivative on the right of (61) is not in action, thus, setting $\delta_1 \cong 0$, we get from (61) the equation $(I_O+lh)\varphi''(t) = -al - Mgl\varphi(t)$, different from the equations in (56), due to the seemingly heavier disk and the additional term $-al$, but with the same initial conditions. This equation exists until $t^* = \delta_2$ at which moment the third derivative in (61) comes into play, changing the right-hand side for $t > \delta_2$ as follows:

$$I_O\varphi'' = -al - Mgl\sin\varphi(t) - l[h\varphi''(t-\delta_1) + k\varphi'''(t-\delta_2)], \quad t > \delta_2. \tag{62}$$

This is the same equation as (61) with all right derivatives. However, the derivative $\varphi'''(.)$ on the right does *not* propel the motion, as it did in Case 3.1, due to a greater delay $\delta_2 > \delta_1$. It adds the additional force $\Delta f(t) = -lk\varphi'''\ (t-\delta_2)$ depending on the rate of change of the actually realized values of the past acceleration $\varphi''\ (t-\delta_2)$ for $t > \delta_2$, assuring a softer rate of change in acceleration which is good for a vehicle and for the people in the vehicle, if we consider in place of the pendulum a swing with people at entertainment centers.

N.B. In the theory of DDEs, functions with delays on the right-hand sides must be defined prior to the start of the motion. For example, to define a unique solution in (61) for $t \geq 0$, cf. (56), the theory requires to define $Q^*(.)$ over the prior segment $[-\delta, 0]$, where $\delta = max(\delta_1, \delta_2)$. With time delays due to information transmittal [18], delayed terms in forces $Q^*(.)$ cannot be "defined" on prior intervals because they *physically do not exist* in those time intervals. Setting them at zero may bring contradictions. Indeed, if $\delta_1 < \delta_2$ and we set $\varphi''^- = \varphi'''^- \equiv 0$ over $[0, \delta_1)$ with $\varphi(0) = \varphi_0 > 0$, then by continuity (Lemma 5.1), we have also $\varphi'' \equiv 0$ on the left in (61), so that at $t = 0$ we get in (61): $0 = -al - Mgl\sin\varphi_0 < 0$, an absurdity. For these reasons, we do not mention prior segments of definition for delayed terms which can be dealt with as they come into action.

3.3. The absence of time delays in mathematical descriptions of motion may lead to substantial errors, especially for small particles at high velocities. In deterministic consideration, this can be seen on the example of a linear harmonic oscillator by comparison of the magnitude of its period with the order of natural time delays. Suppose that gravitation acts on the electron in the same way as on a metal pendulum and that it is added to other forces according to the parallelogram rule. Then we can imagine that small oscillations are superimposed on the rotational motion of an electron around the nucleus which would distort its uniform rotation. In the oscillatory part of the motion along the bottom arc $2\varphi_0$, we can consider the electron as a point-wise mass, so that the second equation in (56) with $r = 0$, for small φ_0 takes the form $\varphi'' + g\varphi/l = 0$, irrespective of the mass of the electron, and the solution is $\varphi = \varphi_0 \sin\omega t$, where $\omega^2 = g/l$, with the period $T = 2\pi(l/g)^{0.5}$. If we take $l = a_0 = 0.529 \times 10^{-8}$ cm which is the radius of the first (innermost) Bohr orbit in the hydrogen atom (Bohr radius), then we have $T = 1.460 \times 10^{-5}$ s. This is just in the middle of the time uncertainty segment for delays δ_1, δ_2 within $[10^{-10}, 1]$ s considered above, so that model (56) is inapplicable to the study of harmonic oscillations of the electron in the hydrogen atom. It means that time delays should be taken into account in deterministic and statistical considerations, especially if computations are involved in the experiment, or particles move in a field of controlled forces, in which cases time delays due to information transmittal really take place.

12. Conclusions

In this paper, the mathematical notion of *left* time-derivatives and its application in physics, mechanics and control of motion is investigated. If a smooth function $f(t, x, y, z)$ is defined over an *open* set of points (t, x, y, z), the partial derivatives with respect to each argument are well defined as the right or left limits which are equal within that set. However, the variable t, usually called *time*, has physical orientation in the sense that it is continuously and uniformly increasing in the positive direction: $t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots$. If the function $f(t, x, y, z)$ is related to a physical process and one of its parameters, say z, is changing with time, then this parameter has the same orientation as time, although with increasing time, $z(t)$ can increase or decrease or stay constant. If we consider a *process* $f(t, x, y, z)$ at certain time moment $t = t^*$, then for $t > t^*$ the values of $x(t), y(t), z(t)$ and $f(.)$ are not yet in existence, thus unknown, so that *right* partial derivatives at $t = t^*$ for *all* parameters t, x, y, z related to the values at $t > t^*$ do *not* exist, and the only partial derivatives that physically exist at $t = t^*$ are *left* (with respect to time orientation) partial derivatives $\partial f^-/\partial t, \ldots, \partial f^-/\partial z$ that correspond to the interval $(t^* - \epsilon, t^*)$, $\epsilon > 0$. Since information transmission takes time, those *left* derivatives are measured and acting with delays. If the process is *observed* by means of some signals (rays of light in [9]), then relativistic effects should be considered for the *image* of the process or motion. These are the fundamental differences between the notions of derivative in mathematics and in physics, mechanics, engineering, life sciences, and technology.

The notion of the left time or partial derivative provides a better understanding of physical processes which may depend on the *left higher order* derivatives, cf. Weber's electro-dynamic law of attraction, and opens new possibilities and ways to control the motion by the use of such derivatives in control systems together with conventional controls, to check them or replace in case of a failure of outboard measuring devices. This goal is achieved in this research by the inclusion of the left higher order derivatives into the active forces of Newton's equations of motion, into the generalized forces of the Lagrange and Hamilton equations, by consideration of the *left* and *delayed higher order* derivatives in the fundamental equations of theoretical physics, and by the application of the left and delayed higher order derivatives to the reactive motion of spacecrafts and in the computerized autopilots of airplanes to assure their safety in the event of malfunction or failure of the outboard velocity sensors, *Pitot tubes*, currently used in aviation.

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